Federal State Autonomous Educational Institution
of Higher Education
«Moscow Institute of Physics and Technology
(National Research University)»

A. E. Umnov, E. A. Umnov

# PARAMETRIC METHODS IN OPERATIONS RESEARCH

Study guide for the Subject

MOSCOW
MIPT, Phystech
November 21, 2026

**Umnov, Alexander Evgenievich,**
**Umnov, Egor Aleksandrovich**

The purpose of the guide is to acquaint the reader with methods for solving operations research problems, the formulations of which contain parameters. First of all, with algorithms for solving conditional optimization problems with constraints of the «inequality» type.

The questions considered in the guide correspond to the standard curricula of higher educational institutions in the subjects «Optimization Methods» and «Operations Research».

The guide can also serve as optional material for a more detailed study of both the theory and applications of these disciplines.

It is assumed that the reader is familiar with the basics of mathematical analysis and linear algebra. The text of the guide includes a description of the formulations and methods for solving a number of problems that are fundamentally important for successful mastering of the course.

The guide is intended for students and postgraduates in the mathematical, technical and economic fields of study.

# Contents

**Introduction** **4**

**1. Construction of feedback functions** **11**
1.1. Theory and algorithms used . . . . . 11
1.1.1. Nonlinear programming problem . . . . . 11
1.1.2. Optimality conditions in a nonlinear programming problem . . . . . 12
1.1.3. Method of smooth penalty functions . . . . . 13
1.2. Feedback functions for linear problems . . . . . 23
1.3. Feedback functions in nonlinear problems . . . . . 32

**2. Feedback Function Properties** **38**
2.1. Convergence of the Feedback Function Method . . . . . 38
2.2. Feedback functions in the linear case . . . . . 47
2.3. Smoothing property of feedback functions . . . . . 66
2.4. Linear extrapolation in the feedback function method . . . . . 72

**3. Feedback functions in extremum search problems** **82**
3.1. Global extremum of functions . . . . . 82
3.2. Maximum function . . . . . 88
3.3. Finding minimax and maximin on a discrete set . . . . . 103
3.4. Problems for multicriteria models . . . . . 106
3.4.1. Method of feedback functions for solving optimization problems for multicriteria models . . . . . 108

3.4.2. An example of a three-level problem for a multicriterial model . . . . . 112
3.5. Game theory problems . . . . . 120

**4. Development of the method of feedback functions 130**
4.1. Analysis of the convergence of the method . . . . . 130
4.2. Modification of feedback functions . . . . . 135
4.2.1. Alternative types of feedback functions . . . . . 135
4.2.2. Reducing consumption of computational resources . 139
4.3. Feedback functions in the parametric linearization method . . . . . 141
4.4. Problems for systems of mathematical models . . . . . 151

**5. Appendix 168**
5.1. Method of parametric linearization. Solution of the lower-level problem . . . . . 168
5.2. Method of parametric linearization. Solution of the upper-level problem . . . . . 176

**References 179**

# Introduction

For any mathematical problem, the desire to simplify its formulation is quite natural. However, in some cases, to obtain a solution to the problem, its formulation has to be generalized or even complicated.

Generalization of the formulation is usually necessary for problems containing some ambiguity in the condition. Complication may be necessary if the solution to the problem requires an additional degree of freedom that is absent from the condition.

Quite often, both to describe the ambiguity and to create an additional degree of freedom in the problem formulation, *parameters* are used.

The purpose of this guide is to study special classes of parametric problems and methods for solving them. Therefore, we will first clarify the concept of a parameter and explain it with examples.

| | |
|---|---|
| `Definition 0.1.1` | *By parameter* in the problem statement we mean a mathematical object that is not uniquely defined *a priori*, but is an admissible element of some given set. |

A brief overview of different types of parametric problems and methods will begin with an obvious example. Suppose that it is required

$$\textit{find real solutions to the equation} \quad x^2 - 6x - 5 = 0\,.$$

This problem can be solved by factoring the left side of the equation or, simply, by selecting roots using Vieta's formula.

However, if the form of the equation is generalized to $x^2 + px + q = 0$, where $p, q \in \mathbb{R}$, then we can use the formula for its roots for arbitrary values of the parameters $p$ and $q$.

It is clear that, if we are interested in the roots of only the original equation, then such a complication of the condition is unlikely to always be useful.

Let's consider another example:

*it is required to select the maximum from the numbers* $x_1 = 4,\ x_2 = -5,\ x_3 = 1.$

A complete enumeration will obviously give its solution $x_{\max} = 4$. But, if for some reason we do not want to apply this method, then the solution can be found,[1] for example like this:

$$x_{\max} = \lim_{\tau \to +0} \tau \ln \left( e^{\frac{x_1}{\tau}} + e^{\frac{x_2}{\tau}} + e^{\frac{x_3}{\tau}} \right) = \lim_{\tau \to +0} \tau \ln \left( e^{\frac{4}{\tau}} + e^{\frac{-5}{\tau}} + e^{\frac{1}{\tau}} \right) = 4. \tag{0.1.1}$$

Formula (0.1.1) uses an auxiliary positive parameter $\tau$. A limit transition to zero is performed for this parameter. Therefore, the value of $\tau$ does not affect the answer to the problem.

Formula (0.1.1) is, of course, more complicated than enumeration of options, but it does not require the logical operation «*if..., then..., otherwise...*»

These two examples allow us to give a more precise

| `Definition 0.1.2` | There are two kinds of parameters:<br>— *exogenous*, which are for the problem being solved the carrier of «external» information, and<br>— *instrumental*, not affecting the result, but necessary for the implementation of the solution method. |
|---|---|

It is clear that both types of parameters can be used simultaneously in one problem. Thus, in the second example, the exogenous parameters are $x_k \quad k = \overline{1,3}$. The parameter $\tau$ is obviously instrumental here.

Let's consider a more complex problem:

*find the value of the Dirichlet integral*

$$I(\alpha) = \int_0^{+\infty} \frac{\sin \alpha x}{x}\, dx, \tag{0.1.2}$$

*where $\alpha$ is an arbitrary real number.*

It is impossible to calculate this integral using the Newton-Leibniz formula since the indefinite integral $\int \frac{\sin \alpha x}{x}\, dx$ cannot be expressed in terms of

[1] Use a calculator to estimate the error in the value $x_{\max}$ that occurs if in formula (0.1.1) the limit is replaced by the value of the function $\tau \ln \left( e^{\frac{4}{\tau}} + e^{\frac{-5}{\tau}} + e^{\frac{1}{\tau}} \right)$, say, at $\tau = 0.1$.

elementary functions. However, by the Dirichlet criterion, the improper integral (0.1.2) converges, that is, the function $I(\alpha)$ has finite values $\forall \alpha \in R$.

These values can be found by using the auxiliary integral

$$\Phi(\alpha, \beta) = \int\limits_0^{+\infty} e^{-\beta x} \frac{\sin \alpha x}{x} dx$$

with a real instrumental parameter $\beta \in [0, 1]$.

The integral is equal to zero at $\alpha = 0$. For $\alpha \neq 0$ it also converges by the Dirichlet criterion for any fixed $\beta \in [0, 1]$. Note also that the integral of *derivative of the integrand with respect to* $\alpha$, has the form

$$\int\limits_0^{+\infty} e^{-\beta x} \cos \alpha x \, dx \, . \tag{0.1.3}$$

It will converge by the Weierstrass criterion *uniformly* on the set $\beta \in (0, 1]$ to $\Phi'_\alpha(\alpha, \beta)$.

The indefinite integral $\int e^{-\beta x} \cos \alpha x \, dx$ is found by double integration by parts. Consequently, the improper integral (0.1.3) can be calculated using the Newton-Leibniz formula. It equals to $\dfrac{\beta}{\alpha^2 + \beta^2}$ (check it yourself).

Now we can find $\Phi(\alpha, \beta)$, integrating the equation $\dfrac{\partial \Phi}{\partial \alpha} = \dfrac{\beta}{\alpha^2 + \beta^2}$ with respect to $\alpha$. As a result, we get $\Phi(\alpha, \beta) = \operatorname{arctg} \dfrac{\alpha}{\beta} + C(\beta)$, where $C(\beta) \equiv 0$ due to $\Phi(0, \beta) = 0 \quad \forall \beta \in [0, 1]$.

Passing to the limit $\beta \to +0$ in the last formula, for a fixed $\alpha > 0$ we find $I(\alpha) = \lim\limits_{\beta \to +0} \Phi(\alpha, \beta) = \dfrac{\pi}{2}$. Finally, due to the oddness of the sine, for the Dirichlet integral for any $\alpha$ we have $I(\alpha) = \dfrac{\pi}{2} \operatorname{sgn} \alpha \, .$

Note that in the problem under consideration the parameter $\alpha$ is exogenous, while the parameter $\beta$ is clearly instrumental.

The parametric approach may also be useful in solving geometric problems. An example of this is the parametrization of the description of families (bundles) of second-order lines on a plane.

Indeed, it is known that any linear combination of equations of second-order lines is an equation of a line of order *is not higher than* 2.

Then we can expect that for some members of the family of lines will be straight lines. That is, they will have equations simpler than the original one. Accordingly, the search for a second-order line satisfying some conditions will also be simplified.

Algorithms in which some variables of the original problem are transformed into parameters can also be classified as parametric. There is a chance that as a result of such a transformation, the problem will be simplified. Say, it will become parametric linear.

An example of such a method is the classical algorithm for finding eigenvalues and eigenvectors of a linear transformation.

According to the definition of an eigenvalue $\lambda$ for a linear transformation $\widehat{A}$ in a finite-dimensional linear space $\Lambda^n$ there is the equality

$$\|\widehat{A} - \lambda\widehat{E}\|\|f\| = \|o\|\,. \tag{0.1.4}$$

Here the transformation $\widehat{E}$ is unitary, and $\|f\| = \|\,\varphi_1\,\varphi_2\,\ldots\,\varphi_n\,\|^{\mathrm{T}} \in \Lambda^n$ is a column of the eigenvector of the transformation $\widehat{A}$.

Equality (0.1.4) is a *nonlinear* system of $n$ equations with unknowns $\{\,\varphi_1,\,\varphi_2,\,\ldots,\,\varphi_n,\,\lambda\,\}$. However, if the unknown eigenvalue $\lambda$ is transformed into a parameter, then equality (0.1.4) turns out to be a *linear homogeneous* system of $n$ equations with $n$ unknowns $\{\,\varphi_1,\ \varphi_2,\,\ldots,\,\varphi_n\,\}$.

The condition for the existence of a *nonzero* solution for such a system is a characteristic equality of the form $\det\|\widehat{A} - \lambda\widehat{E}\| = 0$. Having solved it, we obtain the desired eigenvalues of $\lambda$. Then from (0.1.4) we find for each $\lambda$ the eigenvectors $\|f\|$.

According to the authors, in a review of parametric algorithms, one cannot fail to mention the *penalty function method*. The method can be used to solve some optimization problems.

In this method, the idea of which belongs to Courant [3], *instrumental* parameter is used to transform the original problem of finding a conditional extremum into a parametric problem without constraints.

Let us explain the essence of the penalty function method using the following problem as an example.

*Find the maximum of the function* $\quad F(x) = 2x$
*subject to* $\quad x \le 3.$

We will construct an auxiliary function $A(\tau, x)$, whose values differ little from $F(x)$ for admissible $x$ and are significantly less than $F(x)$ for inadmissible $x$.

For example, such a function can be obtained by adding to the objective function $2x$ a term of the form $-\frac{1}{2\tau}\Big(|x-3|+x-3\Big)^2$, where $\tau$ is a small positive parameter. Then the function $A(\tau,x)$ is

$$A(\tau,x)=2x-\tfrac{1}{2\tau}\Big(|x-3|+x-3\Big)^2.$$

The added term can be considered as a «penalty» for violating the condition $x\le 3$.

The function $A(\tau,x)$ is defined and continuously differentiable on the entire real axis. Its graphs for various values of $\tau$ are shown in Fig. 0.0.1.

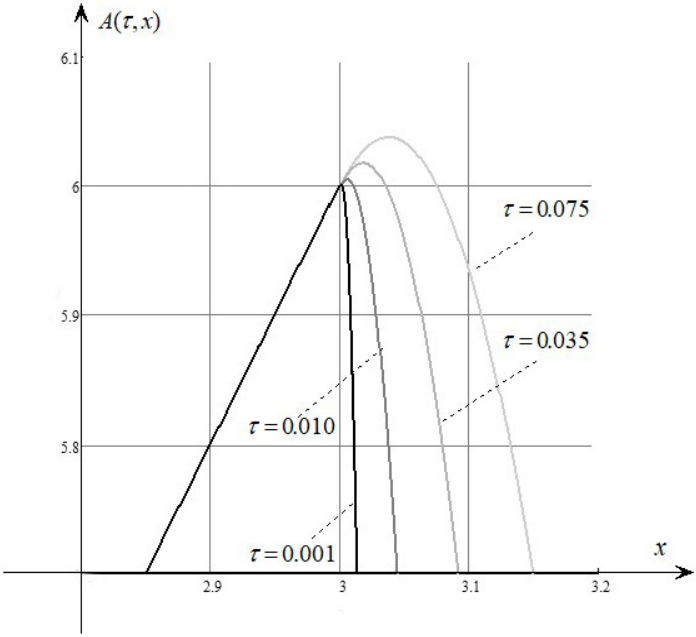



Fig. 0.0.1. Graphs of the function $A(\tau,x)$

It is easy to verify that $A(\tau,x)$ for any fixed positive value of the instrumental parameter $\tau$ has a maximum at the point $\tilde{x}(\tau)=3+2\tau$, equal to $A(\tau,\tilde{x}(\tau))=6+2\tau$.

This extremum is an approximate solution of the original problem. It can be shown that the exact solution to the problem is given by

$$x^*=\lim_{\tau\to+0}\tilde{x}(\tau)=3 \qquad \text{and} \qquad F^*=\lim_{\tau\to+0}A(\tau,\tilde{x}(\tau))=6\,.$$

The penalty function method can be used to solve various types of extremal problems. For example, let us turn once again to the problem of choosing the maximum number in a set $x_1 = 4$, $x_2 = -5$, $x_3 = 1$.

This problem is equivalent to the linear programming problem:

$$\begin{aligned} &\textit{maximize with respect to } v \in E^1 \textit{ the function } F(v) = -v \\ &\textit{subject to} \quad 4 - v \le 0, \quad -5 - v \le 0, \quad 1 - v \le 0. \end{aligned}$$

We will solve this problem using the penalty method, using the function $-\tau e^{\frac{s}{\tau}}$ as a penalty for violating the condition $s \le 0$. The auxiliary function $A(\tau, v)$ in this case will be

$$A(\tau, v) = -v - \tau e^{\frac{4-v}{\tau}} - \tau e^{\frac{-5-v}{\tau}} - \tau e^{\frac{1-v}{\tau}}.$$

We can see that from the equality to zero of the derivative $A(\tau, v)$ with respect to the variable $v$

$$-1 + e^{\frac{4-\overline{v}}{\tau}} + e^{\frac{-5-\overline{v}}{\tau}} + e^{\frac{1-\overline{v}}{\tau}} = 0$$

it follows

$$\overline{v}(\tau) = \tau \ln \left( e^{\frac{4}{\tau}} + e^{\frac{-5}{\tau}} + e^{\frac{1}{\tau}} \right).$$

Whence formula (0.1.1) is obtained.

For the sake of completeness, we note that there are also more complex schemes of instrumental parameterization. For example, algorithms that use several instrumental parameters simultaneously. These include, for example, the linear extrapolation procedure with *vector* instrumental parameters, discussed further in § 2.4 of the guide.

The examples considered above allow us to state that, the use of parameters leads to a more complex formulation of the problem. But the additional degree of freedom that arises in this case can be used to solve a number of problems:

— to construct algorithms for searching for their solutions,

— to analyze the properties of solutions or to search for solutions with special properties.

To demonstrate the validity of this statement, in this guide we will consider in detail a parametric algorithm, hereinafter referred to as the *method of feedback functions*. This approach allows us to find and analyze solutions to various optimization problems containing exogenous parameters.

The main one is the parametric problem [2] of finding the extremum of a function of several variables with inequality constraints

$$\begin{array}{l} \textit{maximize the function} \qquad F(X,V) \quad \textit{by } X \in E^n \\ \textit{subject to:} \quad x \in \Omega \equiv \left\{ \; x \; \middle| \; f_i(X,V) \le 0 \quad i = \overline{1,m} \right\}, \end{array} \tag{0.1.5}$$

where the vector of *exogenous* parameters $V$ is fixed and belongs to some compact domain $\Upsilon \subseteq E^K$.

Let the functions $F(X,V)$ and $f_i(X,V)$ $i = \overline{1,m}$ be twice continuously differentiable with respect to all arguments in their domain of definition $\Theta \subseteq E^n \otimes E^K$. We will consider the solution of problem (0.1.5) to be *the dependence* $\underset{x \in \Omega}{\operatorname{argmax}} F(X,V)$ on $V$, denoted further as $X_v^*$.

Note that the *inequality* type constraints in the formulation of problem (0.1.5) are the reason for the following properties of the $X_v^*$ dependence:

1°. The domain of definition of this dependence may be *ńarrower* than $\Theta$.

2°. The $X_v^*$ dependence may be *ambiguous* and, therefore, not be a function.

3°. If the dependence $X_v^*$ is still a function, it may turn out to be *non-differentiable* with respect to the components of $V$.

It is clear that the noted features limit the applicability of classical methods of differential calculus for finding and studying the properties of $X_v^*$.

Algorithms of *non-differentiable optimization* [2, 6, 13] are usually recommended for these purposes. In this guide, we will consider an alternative approach, which allows us to solve and study both problems (0.1.5) and those reduced to them, using only standard Taylor formulas.

The idea of the proposed approach is to construct a special *smooth approximation* of the $X_v^*$ dependence, which not only preserves the basic properties of the $X_v^*$ dependence, but is also in $E^K$ *a smooth function* with a domain of definition *not ńarrower* than $\Theta$.

The basis of the proposed method is the joint use of an instrumental parameter and some special functions, which implement feedback for the variables of the problem solved and its Lagrange multipliers.

[2] This problem is sometimes called the *parametric programming problem.*

# Chapter 1.

# Construction of feedback functions

This chapter describes feedback functions and provides simple examples of their use.

Formally, feedback functions are introduced axiomatically. In practice, however, their definition can be arrived at by investigating the procedure for solving a mathematical programming problem by the penalty function method.

Therefore, it seems appropriate to begin the text of the guide with a consideration of some issues from the theory of mathematical programming. This knowledge will be useful both for substantiating the feedback function method itself, and for solving various types of parametric problems with its help.

## 1.1. Theory and algorithms used

### 1.1.1. Nonlinear programming problem

Let $X \in E^n$ have coordinate column $\|X\| = \| x_1\, x_2\, \ldots\, x_n \|^{\mathrm{T}}$. We will call a mathematical programming problem a task of the form

$$\begin{aligned} &\textit{maximize with respect to } X \quad \textit{objective function } F(X) \\ &\textit{subject to conditions:} \\ &\qquad X \in \Omega, \quad \textit{where } \Omega \equiv \left\{ \; X \;\middle|\; f_i(X) \le 0 \quad i = \overline{1,m} \; \right\}. \end{aligned} \tag{1.1.1}$$

Suppose also that problem (1.1.1) has a finite local solution $F^* \stackrel{def}{=} F(X^*)$ at $X^*$ consisting, perhaps, of more than one point. Also let each of the functions $F(X)$, $f_i(X)$ $i = \overline{1,m}$ be twice continuously differentiable with respect to all its arguments.

Finally we will use

| | |
|---|---|
| `Definition 1.1.1` | The constraint $f(X) \le 0$ is called *active* at the point $X_0 \in E^n$ , if $f(X_0) \ge 0$ and, accordingly, is called *inactive* if $f(X_0) < 0$. |

## 1.1.2. Optimality conditions in a nonlinear programming problem

Let for problem (1.1.1) there exist a regular Lagrange function of the form

$$L(X, \Lambda) = F(X) - \sum_{i=1}^{m} \lambda_i f_i(X)\,,$$

where $\Lambda \in E^m$ is a vector having a coordinate column

$$\|\Lambda\| = \|\, \lambda_1\, \lambda_2\, \ldots\, \lambda_m \,\|^{\mathrm{T}},$$

and $X^*$ is a solution to problem (1.1.1). The numbers $\lambda_i$ $i = \overline{1,m}$ are usually called *the Lagrange multipliers* of problem (1.1.1).

A necessary condition for the solvability of problem (1.1.1) is given by

`Theorem 1.1.1` (*Karush-Kuhn-Tucker*)

**Let $X^*$ be a solution of problem (1.1.1), then there exists a vector $\Lambda^*$ such that**

$$\operatorname*{grad}_x L(X^*, \Lambda^*) = o,$$

**and for each $i = \overline{1,m}$ the conditions are valid**

$$\lambda_i^* \ge 0 \quad \textbf{and} \quad \lambda_i^* f_i(X^*) = 0\,.$$

The last statement in Theorem 1.1.1 is usually called the conditions of *complementary slackness*. The proof of this theorem can be found in various sources, for example, in [4].

We also note the following.

1. The Lagrange function and multipliers are defined only for problem (1.1.1), which has a finite solution $F^*$.
2. If the constraint $f_i(X) \le 0$ is inactive in $X^*$, then $\lambda_i^* = 0$. The converse is not true. In other words, Theorem 1.1.1 allows the case when the equalities $f_i(X^*) = 0$ and $\lambda_i^* = 0$ are satisfied simultaneously for some $i = \overline{1,m}$.

### 1.1.3. Method of smooth penalty functions

We apply a special version of the method of *smooth* penalty functions [3, 29] to solve problem (1.1.1). This version consists of sequential maximization with respect to $X \in E^n$ of the auxiliary function

$$A(\tau, X) = F(X) - \sum_{i=1}^{m} P\Big(\tau, f_i(X)\Big) \tag{1.1.2}$$

for positive values of the instrumental parameter $\tau$ tending to zero.

It is assumed that in (1.1.2) the function $P(\tau, s)$, «penalizing» violation of the constraint $s \le 0$, satisfies the conditions:

1°. It is defined $\forall \tau > 0$ and $\forall s$, and also has continuous partial derivatives with respect to $s$ up to and including the second order.

2°. For all $\tau > 0$ and $\forall s$ the following inequalities hold:

$$P(\tau, s) > 0\,, \qquad \frac{\partial P}{\partial s}(\tau, s) > 0\,, \qquad \frac{\partial^2 P}{\partial\, s^2}(\tau, s) > 0\,. \tag{1.1.3}$$

3°.

$$\lim_{\tau \to +0} P(\tau, s) = \begin{cases} +\infty & \text{if } s > 0\,, \\ 0 & \text{for } s \le 0\,. \end{cases}$$

4°. Let, *in addition*, the function $P(\tau, s)$ have the form

$$P(\tau, s) = \tau\, \Phi\left(\frac{s}{\tau}\right), \tag{1.1.4}$$

where $\Phi$ is a function of one variable. That is, $\Phi$ depends only on the ratio $\dfrac{s}{\tau}$.

An example of such a function is $P(\tau, s) = \tau \exp\left(\dfrac{s}{\tau}\right)$. However, neither the barrier logarithmic function nor the standard quadratic penalty function[1] the set of conditions 1°—4° do not satisfy.

Note also that by 2° on the sets $s \in [\delta_0, +\infty)\ \forall \delta_0 > 0$ and $s \in (-\infty, 0\,]$ convergence in 3° is *uniform*.

To simplify the text, we will use the notations $S\big(\tau, X\big) = \sum\limits_{i=1}^{m} P\Big(\tau, f_i(X)\Big)$ and, accordingly,

$$A(\tau, X) = F(X) - S\big(\tau, X\big)\,.$$

According to the assumptions made, the function $A(\tau, X)$ is defined $\forall \tau > 0$ and $\forall X \in \Omega$. In addition, this function is twice continuously differentiable with respect to the components of $X$.

Let[2] $\tilde{X}_\tau$ is a stationary point of function (1.1.2), defined by the equation

$$\operatorname*{grad}_x A\big(\tau, \tilde{X}_\tau\big) = o. \tag{1.1.5}$$

Then the essence of the penalty function method can be formulated as the equality

$$\lim_{\tau \to +0} A\big(\tau, \tilde{X}_\tau\big) = F(X^*)\,.$$

Moreover, if the solution of problem (1.1.1) is locally unique, then the mapping $\tau \to \tilde{X}_\tau$ is unique and the relations

$$x_j^* = \lim_{\tau \to +0} \tilde{x}_{\tau j} \qquad j = \overline{1, n}$$

are valid.

Because of the importance of this version of the penalty function method for further discussions, we will consider its justification in detail.

---

[1] That is, a function of the form $P(\tau, s) = \dfrac{(|s| + s)^2}{8\tau} = \begin{cases} \dfrac{s^2}{2\tau} & \text{for} \quad s \geq 0, \\ 0 & \text{for} \quad s < 0\,. \end{cases}$

[2] A notation of the form $A_b$, where $A$ and $b$ are *real* variables, will be used in cases where $A$ depends on $b$ , but this dependence is not necessarily a *function*. For functional dependencies, we retain the standard notation $A(b)$.

Suppose that there exists $\tau_0 > 0$ such that the function $A(\tau, X)$ for any fixed $\tau \in (0, \tau_0)$ is continuous on *compact set*[3] $C$ containing $X^*$. In this case $A(\tau, X)$ reaches its local *maximum* on the set $C$ with respect to $X$ in $\tilde{X}_\tau = \underset{x \in C}{\operatorname{argmax}} A(\tau, X)$.

Let $\Omega_\delta$ be *$\delta$-neighborhood of the set* $\Omega$, defined $\forall \delta > 0$ by the system of inequalities

$$f_i(X) \le \delta \qquad i = \overline{1, m}\,. \tag{1.1.6}$$

Denote $\omega \stackrel{def}{=} \Omega_\delta \cap C$ and show that the following theorem holds.

`Theorem 1.1.2` **The equality**

$$\lim_{\tau \to +0} \max_{x \in \Omega} A(\tau, X) = F^* \tag{1.1.7}$$

**are valid.**

`Proof.`

1°. Let the conditions of problem (1.1.1) and the assumptions about the properties of the function $P(\tau, s)$ be satisfied.

We use the continuity of the functions $P\big(\tau, f_i(X)\big)$ $i = \overline{1, m}$ with respect to $X$ for $X \in \omega$ for fixed $\tau \in (0, \tau_0)$. Let us show that in this case $\forall \varepsilon > 0$ there be $\delta = \delta_\varepsilon > 0$ and $\tau = \tau_\varepsilon > 0$ such that $\max\limits_{x \in \omega} A(\tau_\varepsilon, X) \ge \max\limits_{x \in \omega} F(X) - \varepsilon$.

Now we will describe a method for choosing such $\delta_\varepsilon$ and $\tau_\varepsilon$ for given $\varepsilon$.

Let us first estimate the value

$$P\big(\tau_\varepsilon, f_i(X)\big) = \tau_\varepsilon\, \Phi\left(\frac{f_i(X)}{\tau_\varepsilon}\right)$$

in the set $\omega$. This function increases monotonically in $f_i(X)$. Therefore, if for some $\delta_{(i)\varepsilon} > 0$ take $\tau_\varepsilon = \delta_{(i)\varepsilon}$, then by virtue of (1.1.6) we have

$$\tau_\varepsilon \Phi\left(\frac{f_i(X)}{\tau_\varepsilon}\right) \le \tau_\varepsilon \Phi\left(\frac{\delta_{(i)\varepsilon}}{\tau_\varepsilon}\right) = \delta_{(i)\varepsilon}\Phi(1) \qquad i = \overline{1, m}\,.$$

[3] As $C$ we can take a sufficiently large closed ball (or parallelepiped) in $E^n$. The existence of such a $C$, as a rule, follows from the meaningful interpretation of problem (1.1.1).

Let's put $\delta_\varepsilon = \max\limits_{i=\overline{1,m}}\left\{\delta_{(i)\varepsilon}\right\}$. Then

$$\max_{x\in\omega} A(\tau_\varepsilon, X) = \max_{x\in\omega}\left(F(X) - \sum_{i=1}^{m} P\big(\tau, f_i(X)\big)\right) \geq$$

$$\geq \max_{x\in\omega} F(X) - m\delta_\varepsilon\Phi(1)\,.$$

So, if $\forall\varepsilon > 0$ choose $\quad \delta_\varepsilon = \tau_\varepsilon = \dfrac{\varepsilon}{m\,\Phi(1)}$, then

$$\max_{x\in\omega} A(\tau_\varepsilon, X) \geq \max_{x\in\omega} F(X) - \varepsilon\,. \tag{1.1.8}$$

If it turns out that $\delta_\varepsilon > \tau_0$, then we take $\delta_\varepsilon = \tau_0$. This guarantees the fulfillment of (1.1.8).

On the other hand, from the definition of $P(\tau, s)$ follows the inequality $\max\limits_{x\in\omega} S(\tau, X) \geq 0$. Then for any positive sequence $\{\tau_{(k)\varepsilon}\}$ by (1.1.8) we obtain

$$\max_{x\in\omega} F(X) \geq \max_{x\in\omega} A(\tau_{(k)\varepsilon}, X) \geq \max_{x\in\omega} F(X) - \varepsilon\,. \tag{1.1.9}$$

Under the assumptions made $\delta_\varepsilon \to 0$ and $\Omega_{\delta_\varepsilon} \to \Omega$ when $\varepsilon \to 0$. Then, using the definition of the limit of a function according to Heine and its properties, from (1.1.9) we obtain the equality to be proved $\quad \lim_{\tau\to+0} \max\limits_{x\in\Omega} A(\tau, X) = F^*$.

`The theorem is proved.`

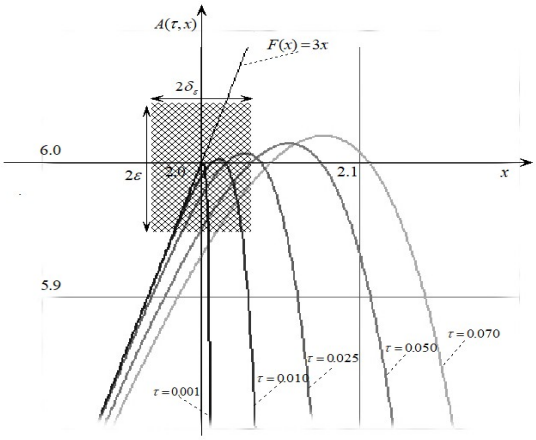



Fig. 1.1.1. To the proof of Theorem 1.1.2

We will illustrate the procedure for choosing the values of $\delta_\varepsilon$ and $\tau_\varepsilon$ graphic for the problem:

*find the maximum of the function* $F(x) = 3x$
*subject to* $x \leq 2$.

Let the auxiliary function used to solve this problem have the form

$$A(\tau, x) = 3x - \tau \exp\left(\frac{x-2}{\tau}\right).$$

Figure 1.1.1 shows that by changing the instrumental parameter $\tau$ it is possible to achieve the local maximum of the auxiliary function $A(\tau, X)$ to fall inside the shaded region. This region in $E^{n+1}$ is a cylinder of radius $\delta_\varepsilon$ and height $2\varepsilon$.

For practical search of elements of the set $X^*$ by the penalty function method it turns out to be useful

`Theorem 1.1.3` **Let the sequence $\{Y_k\}$ be contained in the set $\omega$ and there exist bounded positive numerical sequences $\{\alpha_k\}$ and $\{\tau_k\}$ such that**

$$A(\tau_k, Y_k) \geq \max_{y \in \omega} A(\tau_k, Y) - \alpha_k \qquad \forall k \in \mathbb{N},$$

**then any limit point of the sequence $\{Y_k\}$ is a solution to problem (1.1.1).**

`Proof.`

The existence of a limit point $Y^*$ of the sequence $\{Y_k\}$ follows from the compactness of the set $\omega$. Without loss of generality, we can also assume that $\lim\limits_{k \to +\infty} \{Y_k\} = Y^*$.

From the assumption of boundedness of $F(X^*)$ and the properties of the function $P(\tau, s)$ it follows that the limit point $Y^*$ belongs to the set $\omega$. Then estimates

$$F^* = \max_{x \in \omega} F(X) \geq F(Y^*) = \lim_{k \to +\infty} F\big(Y_k\big) \geq \lim_{k \to +\infty} A\big(\tau_k, Y_k\big) \geq$$

$$\geq \lim_{\tau \to +0} \Big(\max_{y \in \omega} A(\tau_k, Y) - \alpha_k\Big) = \max_{x \in \omega} F(X) = F^*$$

are valid.

This means that $F(Y^*) = F(X^*) \quad \Longrightarrow \quad Y^* \in X^*.$

`The theorem is proved.`

It will also be true

`Corollary 1.1.1` **In the case of local uniqueness of the solution of problem (1.1.1)** $\lim_{\tau \to +0} \tilde{x}_j(\tau) = x_j^* \quad j = \overline{1,n}\,.$

`Proof.`

The conclusion of the corollary is obvious, since in this case the set $X^*$ consists of a single point.

`The corollary is proved.`

Theorem 1.1.2 can also be proved in a different way.

Let us recall first, that the operations of taking the supremum (or infimum) for functional sequences, generally speaking, *does not commute* with the operation of passing to the limit.

This fact is illustrated by

`Example 1.1.1.` Let the functional sequences

$$h_n(x) = \begin{cases} 1, & \text{if} \quad x = 0\,, \\ 0, & \text{if} \quad 0 < x \le \dfrac{1}{n}, \\ 1, & \text{if} \quad \dfrac{1}{n} < x \le 2 \end{cases} \quad \text{and} \quad g_n(x) = \begin{cases} 0, & \text{if} \quad x = 0\,, \\ 1, & \text{if} \quad 0 < x \le \dfrac{1}{n}, \\ 0, & \text{if} \quad \dfrac{1}{n} < x \le 2\,. \end{cases}$$

be defined on the set $\Omega = [0,2]$ $\forall n \in \mathbb{N}$.

The sequences $\{h_n(x)\}$ and $\{g_n(x)\}$ converge pointwise $\forall x \in \Omega$ respectively to the limit functions $h^*(x) \equiv 1$ and $g^*(x) \equiv 0$.

For the first sequence we have

$$\lim_{n\to\infty} h_n(x) = h^*(x) \equiv 1 \quad \text{And} \quad \sup_{x\in\Omega} h_n(x) = 1 \qquad \longrightarrow$$

$$\longrightarrow \qquad \sup_{x\in\Omega} \lim_{n\to\infty} h_n(x) = 1 = \lim_{n\to\infty} \sup_{x\in\Omega} h_n(x)\,.$$

While for the second

$$\lim_{n\to\infty} g_n(x) = g^*(x) \equiv 0 \quad \text{and} \quad \sup_{x\in\Omega} g_n(x) = 1 \qquad \longrightarrow$$

$$\longrightarrow \qquad \sup_{x\in\Omega} \lim_{n\to\infty} g_n(x) = 0 \neq 1 = \lim_{n\to\infty} \sup_{x\in\Omega} g_n(x)\,.$$

That is, in the first case, the result does not depend on the order of operations. In the second case, changing the sequence of operations leads to a change in the result.

Let us pay attention to the difference between the cases considered.

> The limit function $h^*(x)$ is an upper bound for the sequence $\{h_n(x)\}$, but $g^*(x)$ is not an upper bound for $\{g_n(x)\}$ .

The proof of the condition for the commutativity of the operations of finding the exact bound and the limit passage is given by

`Lemma 1.1.1` **Let on the set $\Omega$ there be defined a functional sequence $\{\varphi_k(X)\}$, pointwise converging to a function $\Phi^*(X)$ bounded from above, for which**

$$\sup_{X\in\Omega} \varphi^*(X) = \Phi^* < +\infty.$$

**Let also**

$$\varphi_k(X) \leq \varphi^*(X) \qquad \forall X \in \Omega,\ \forall k \in N\,, \quad (1.1.10)$$

**then the equalities are true**

$$\lim_{k\to\infty} \sup_{X\in\Omega} \varphi_k(X) = \sup_{X\in\Omega} \lim_{k\to\infty} \varphi_k(X) = \Phi^*$$

`Proof.`

From condition (1.1.10), the definition of the supremum and pointwise convergence of the sequence $\{\varphi_k(x)\}$ we have

$$\forall \varepsilon > 0 \qquad \exists N_\varepsilon \in N,\ \exists x_\varepsilon \in \Omega$$

such that $\forall n \geq N_\varepsilon$ the following inequalities are true:

$$\varphi^*(x_\varepsilon) > \Phi^* - \frac{\varepsilon}{2}, \qquad \varphi^*(x_\varepsilon) - \varphi_n(x_\varepsilon) < \frac{\varepsilon}{2},$$

Then the following relations are true:

$$0 \leq \Phi^* - \sup_{x\in\Omega} \varphi_n(x) \leq \Phi^* - \varphi_n(x_\varepsilon) < \Big(\varphi^*(x_\varepsilon) + \frac{\varepsilon}{2}\Big) - \varphi_n(x_\varepsilon) =$$

$$= \Big(\varphi^*(x_\varepsilon) - \varphi_n(x_\varepsilon)\Big) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

That is, we got:

$$\forall \varepsilon > 0 \qquad \exists N_\varepsilon \in N \;:\quad \forall n \geq N_\varepsilon \qquad \longrightarrow$$

$$\longrightarrow \qquad \Big| \sup_{x\in\Omega} \varphi_n(x) - \Phi^* \Big| < \varepsilon\,.$$

This is the conclusion of the lemma being proved:

$$\lim_{n\to\infty} \sup_{x\in\Omega} \varphi_n(x) = \Phi^* = \sup_{x\in\Omega} \lim_{n\to\infty} \varphi_n(x)\,.$$

`The lemma is proved.`

In practice, a useful alternative to Lemma 1.1.1 may be

`Lemma 1.1.2` **Let $\varphi(\tau, X)$ be a continuous function on the compact set $\Omega \subseteq E^n$, depending on the parameter $\tau \in (\alpha, \gamma)$, which uniformly converges on $\Omega$ as $\tau \to \beta \in (\alpha, \gamma)$ to the function $\Phi(X)$. Then the equalities are true**

$$\lim_{\tau\to\beta} \max_{X\in\Omega} \varphi(\tau, X) = \max_{X\in\Omega} \lim_{\tau\to\beta} \varphi(\tau, X) = \max_{X\in\Omega} \Phi(X)\,.$$

`Proof.`

Prove Lemma 1.1.2 yourself, using the definition of the limit of a function according to Heine and the conclusion of Lemma 1.1.1.

`The lemma is proved.`

We now bring

An alternative proof of Theorem 1.1.2.

We use the commutativity of the operations maximization with respect to $x \in \omega$ and limit passage $\tau_k \to +0$ for the functional sequence $\{A(\tau_k, x)\}$,

By property 3° (1.1.3) we have

$$\lim_{\tau \to +0} \max_{x \in \omega} A(\tau, x) = \max_{x \in \omega} \lim_{\tau \to +0} A(\tau, x) =$$

$$= \max_{x \in \omega} \lim_{\tau \to +0} \left( F(x) - \sum_{i=1}^{m} P\Big(\tau, f_i(x)\Big) \right) =$$

$$= \max_{x \in \omega} \left( F(x) - \lim_{\tau \to +0} \sum_{i=1}^{m} P\Big(\tau, f_i(x)\Big) \right) =$$

$$= \max_{x \in \omega} F(x) = F^*.$$

The theorem is proved.

The mapping $\tau \to \tilde{X}_\tau$ is implicitly defined by relation (1.1.5). Its important properties are described by the following two theorems.

Theorem 1.1.4 **For problem (1.1.1) under the assumptions given in § 1.1.3 mapping $\tau \to \tilde{X}_\tau \equiv \operatorname{argmax}_{x \in \omega} A(\tau, X)$ is a vector function in $E^n$.**

Proof.

To prove the conclusion it is sufficient to show the non-degeneracy of the Hessian of the function $A(\tau, X)$ for fixed $\tau \in (0, \tau_0]$.

Then from the theorem on the system of implicit functions [17] it follows that on $(0, \tau_0]$ there exists a smooth vector function $\tilde{X}(\tau)$, which will be a solution of equation (1.1.5).

Theorems similar to this one will be considered in detail in Chapter 2. Here we suggest that you perform the proof yourself as an exercise.

The theorem is proved.

Theorem 1.1.5 **For problem (1.1.1), which has unique solution under the conditions of Theorem 1.1.4 and in which the values, $\lambda_i^* \quad i = \overline{1,m}$ are uniquely determined, the equalities hold**

$$\lim_{\tau \to +0} \frac{\partial P}{\partial s}\Big(\tau, f_i\big(\tilde{X}_\tau\big)\Big) = \lambda_i^* \qquad i = \overline{1,m}\,.$$

Proof.

1°. We denote the set of indices of constraints of problem (1.1.1) that are active at point $X^*$ as $J_\Omega$. Then, by point 3° of (1.1.3),

$$\lim_{\tau \to +0} \frac{\partial P}{\partial s}\Big(\tau, f_i\big(\tilde{X}_\tau\big)\Big) = 0 \qquad \forall i \notin J_\Omega\,.$$

2°. Let the regularity condition be satisfied at the point $X^*$

$$\det \left\| \frac{\partial f_i}{\partial x_j} \right\| \neq 0 \qquad i \in J_\Omega\,, \quad j = \overline{1,n}\,.$$

Then the Lagrange multipliers $\lambda_I^*$ are uniquely determined for $i \in J_\Omega^*$. For all $i \notin J_\Omega \;\; \lambda_i^* = 0$.

3°. Coordinate form of the equality $\operatorname*{grad}_x L(X^*, \Lambda^*) = o$ in the Theorem 1.1.1 looks like

$$\frac{\partial F}{\partial x_j}(X^*) - \sum_{i \in J_\Omega} \lambda_i^* \frac{\partial f_i}{\partial x_j}(X^*) = 0 \qquad j = \overline{1,n}\,. \qquad (1.1.11)$$

On the other hand, the stationarity condition of the auxiliary function $A(\tau, X)$ with respect to the variable $X$ at the point $\tilde{X}_\tau$ will be

$$\frac{\partial F}{\partial x_j}(\tilde{X}_\tau) - \sum_{i \in J_\Omega} \frac{\partial P}{\partial s}\Big(\tau, f_i\big(\tilde{X}_\tau\big)\Big) \frac{\partial f_i}{\partial x_j}(\tilde{X}_\tau) = 0 \qquad j = \overline{1,n}\,.$$

Passing in these equalities to the limit at $\tau \to +0$, by virtue of Corollary 1.1.1 and the assumptions made about continuity, we obtain

$$\frac{\partial F}{\partial x_j}(X^*) - \sum_{i \in J_\Omega} \left( \lim_{\tau \to +0} \frac{\partial P}{\partial s}\Big(\tau, f_i\big(\tilde{X}_\tau\big)\Big) \right) \frac{\partial f_i}{\partial x_j}(X^*) = 0$$

$$j = \overline{1, n}\,. \tag{1.1.12}$$

4°. Term-by-term subtraction of equalities (1.1.12) from (1.1.11) gives

$$\sum_{i \in J_\Omega} \left( \lambda_i^* - \lim_{\tau \to +0} \frac{\partial P}{\partial s}\Big(\tau, f_i\big(\tilde{X}_\tau\big)\Big) \right) \frac{\partial f_i}{\partial x_j}(X^*) = 0$$

$$j = \overline{1, n}\,.$$

The resulting set of equalities is a homogeneous system of $n$ linear equations with a non-singular main matrix. The unknowns in this system are the $n$ expressions in large parentheses.

It follows from Cramer's theorem that this system has *only zero* solution. Hence,

$$\lambda_i^* = \lim_{\tau \to +0} \frac{\partial P}{\partial s}\Big(\tau, f_i\big(\tilde{X}_\tau\big)\Big) \qquad i = \overline{1, m}\,.$$

`The theorem is proven.`

## 1.2. Feedback functions for linear problems

If in problem (1.1.1) it is known which of the constraints are active at the solution point, then the search for this solution is simplified. It will be reduced to solving the problem with constraints of the «equality» type. Therefore, it seems advisable to try to solve it using some indicators of constraint activity.

For example, functions whose values are close to zero for active constraints and are not such for inactive ones, can be used as such an indicator. Further such scheme will be proposed. We will call it as *the feedback function method*.

This method is described more simply in the case of linear functions $F(X)$ and $f_i(X)$ $i = \overline{1,m}$. Therefore, we first consider a pair of mutually dual linear programming problems written in *symmetric* form.

Let the vectors $X \in E^n$ and $\Lambda \in E^m$ be, respectively, the desired variables for the following problems.

1. **Primary** linear programming problem:

$$\begin{aligned} &\textit{maximize with respect to } X \textit{ function} \quad F(X) = \sum_{j=1}^{n} \sigma_j x_j \\ &\textit{subject to} \qquad x_j \geq 0 \quad j = \overline{1,n}, \\ &f_i(X) = -\beta_i + \sum_{j=1}^{n} \alpha_{ij} x_j \leq 0 \quad i = \overline{1,m}. \end{aligned} \tag{1.2.1}$$

2. **Dual** linear programming problem:

$$\begin{aligned} &\textit{minimize with respect to } \Lambda \textit{ function} \quad G(\Lambda) = \sum_{i=1}^{m} \beta_i \lambda_i \\ &\textit{subject to} \quad \lambda_i \geq 0 \;\; i = \overline{1,m}, \\ &g_j(\Lambda) = -\sigma_j + \sum_{i=1}^{m} \alpha_{ij} \lambda_i \geq 0 \quad j = \overline{1,n}. \end{aligned} \tag{1.2.2}$$

The solution to the dual problem will be denoted as $\Lambda^*$ and $G^* = G(\Lambda^*)$.

We use the method of smooth penalty functions with auxiliary functions

$$\begin{aligned} A_P(\tau, X) &= F(X) - \sum_{i=1}^{m} P\big(\tau, f_i(X)\big) \; - \sum_{j=1}^{n} P\big(\tau, (-x_j)\big)\,, \\ A_D(\tau, \Lambda) &= G(\Lambda) + \sum_{j=1}^{n} P\big(\tau, -g_j(\Lambda)\big) + \sum_{i=1}^{m} P\big(\tau, (-\lambda_i)\big)\,. \end{aligned} \tag{1.2.3}$$

to solve problems (1.2.1) — (1.2.2).

Stationary points of functions (1.2.3) will obviously depend on the values of the parameter $\tau$. Therefore, we will denote them as $\tilde{X}_\tau$ and $\breve{\Lambda}_\tau$. Then, by virtue of the properties of the function $P(\tau, s)$

$$\operatorname*{grad}_x A_P\big(\tau, X\big)\Big|_{X=\tilde{X}_\tau} = o \qquad \text{and} \qquad \operatorname*{grad}_\lambda A_D\big(\tau, \Lambda\big)\Big|_{\Lambda=\breve{\Lambda}_\tau} = o.$$

According to Theorem 1.1.2, the value $F(\tilde{X}_\tau) = \operatorname*{argmax}_x A_P(\tau, X)$ can serve as an approximation for $F^*$, and $G(\breve{\Lambda}_\tau) = \operatorname*{argmax}_\lambda A_D(\tau, \Lambda)$ can be,

respectively, an approximation of $G^*$. Moreover, by virtue of Theorem 1.1.5 the equalities

$$\begin{aligned} \lambda_i^* &= \lim_{\tau\to+0} \frac{\partial P}{\partial f_i}\Big(\tau, \quad f_i(\tilde{X}_\tau)\Big) \qquad i = \overline{1,m}\,, \\ x_j^* &= \lim_{\tau\to+0} \frac{\partial P}{\partial g_j}\Big(\tau, -g_j(\breve{\Lambda}_\tau)\Big) \qquad j = \overline{1,n}\,. \end{aligned} \tag{1.2.4}$$

are valid.

Here the components of the vector $\Lambda^*$ are Lagrange multipliers for problem (1.2.1), and the components of the vector $X^*$ are Lagrange multipliers in problem (1.2.2).

It is impossible to use relations (1.2.4) together as a system to find estimates of $X^*$ and $\Lambda^*$. Indeed, for each fixed $\tau > 0$ we have

$$\frac{\partial P}{\partial f_i}\Big(\tau, \quad f_i(\tilde{X}_\tau)\Big) \neq \breve{\lambda}_{i\tau} \quad i = \overline{1,m}$$

and

$$\frac{\partial P}{\partial g_j}\Big(\tau, -g_j(\breve{\Lambda}_\tau)\Big) \neq \tilde{x}_{j\tau} \quad j = \overline{1,n}\,.$$

These relations turn out to be true equalities only *in the limit as* $\tau \to +0$.

However, for a fixed $\tau > 0$ equalities (1.2.4) can be *formally combined* into one system of equations with *new* unknowns $\overline{X}_\tau$ and $\overline{\Lambda}_\tau$

$$\begin{cases} \overline{\lambda}_{\tau i} &= \dfrac{\partial P}{\partial s}\Big(\tau, \quad f_i(\overline{X}_\tau)\Big) \qquad i = \overline{1,m}\,, \\ \overline{x}_{\tau j} &= \dfrac{\partial P}{\partial s}\Big(\tau, -g_j(\overline{\Lambda}_\tau)\Big) \qquad j = \overline{1,n}\,, \end{cases} \tag{1.2.6}$$

It is natural to assume that the solutions of system (1.2.6) will also be some approximations of the solutions of problems (1.2.1) and (1.2.2).

Here we note that by virtue of conditions (1.1.3) the function $\dfrac{\partial P}{\partial s}(\tau, s)$ with respect to the variable $s$ has *inverse function* $Q(\tau, s) = \operatorname{inv}\left(\dfrac{\partial P}{\partial s}(\tau, s)\right)$. Therefore, system (1.2.6) can be written as

$$\begin{cases} f_i\big(\tau, \overline{X}_\tau\big) &= Q(\tau, \overline{\lambda}_{i\tau}) \qquad i = \overline{1,m}\,, \\ -g_j\big(\tau, \overline{\Lambda}_\tau\big) &= Q(\tau, \overline{x}_{j\tau}) \qquad j = \overline{1,n} \end{cases}$$

or

$$\begin{cases} -\beta_i + \sum\limits_{j=1}^{n} \alpha_{ij}\overline{x}_j = Q(\tau, \overline{\lambda}_i) \qquad i = \overline{1,m}\,, \\ -\sigma_j + \sum\limits_{i=1}^{m} \alpha_{ij}\overline{\lambda}_i = -Q(\tau, \overline{x}_j) \qquad j = \overline{1,n}\,. \end{cases} \tag{1.2.7}$$

In Chapter 2 it will be proved that system (1.2.7) has positive solutions *for any* linear programming problems.

Here we only note that the function $Q(\tau, s)$ monotonically increases in $s$ and is continuously differentiable for any positive $\tau$ and $s$. Note also that *the positivity of the components of the vectors* $\overline{X}_\tau$ and $\overline{\Lambda}_\tau$ follows from system (1.2.6) and inequalities (1.1.3).

In addition, $Q(\tau, s)$ can be considered as a function implementing *feedback* in the system of constraints of problems (1.2.1) and (1.2.2).

Indeed, by the values of the function $Q(\tau, s)$ it is possible to determine which of the constraints in these problems are active at the solution point, and which are not. Specifically, small in absolute value $Q(\tau, s)$ indicate the activity of the constraint, and large ones indicate its absence.

This justifies the appropriateness of the term «feedback» both when referring to functions of the form $Q(\tau, s)$, and to methods using these functions.

The possible validity of the formulated assumptions is illustrated by the following examples.

`Example 1.2.1.`

**Primary problem**

*maximize in* $E^2$ *the function* $F(X) = 2x_1 + 3x_2$,
*subject to* $x_1 \geq 0\,,\ x_2 \geq 0$
*and*
$x_1 + 2x_2 \leq 6\,,\ \ 2x_1 + x_2 \leq 6\,.$

**Dual problem**

*minimize in* $E^2$ *the function* $G(\Lambda) = 6\lambda_1 + 6\lambda_2$,
*subject to* $\lambda_1 \geq 0\,,\ \lambda_2 \geq 0$
*and*
$\lambda_1 + 2\lambda_2 \geq 2\,,\ \ 2\lambda_1 + \lambda_2 \geq 3\,.$

The solutions of this pair of mutually dual problems will be, respectively,

$$x_1^* = 2,\ x_2^* = 2,\ F^* = 10 \qquad \text{and} \qquad \lambda_1^* = \tfrac{4}{3},\ \lambda_2^* = \tfrac{1}{3},\ G^* = 10\,.$$

These solutions are shown graphically by points $A^*$ and $B^*$ in Fig. 1.2.1.

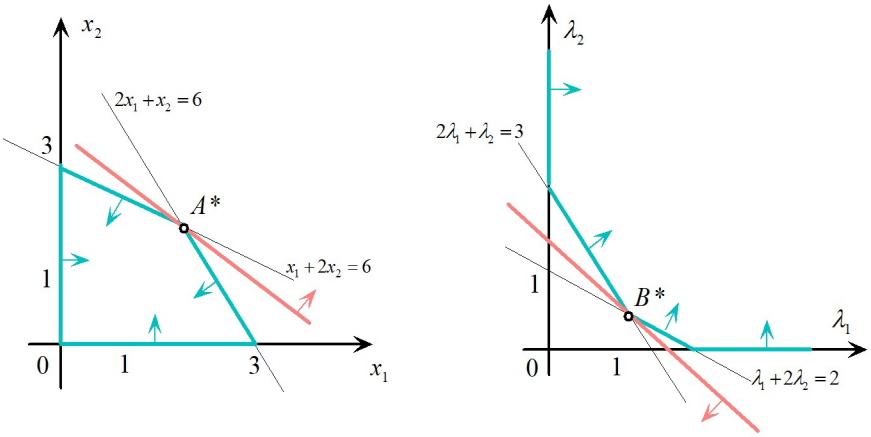



Fig. 1.2.1. Graphical interpretation of the primary and dual problems in example 1.2.1

To use the feedback function method, it is necessary to choose a specific function $Q(\tau, s)$.

This choice is limited only by the conditions specified in § 1.1.3, which the function $P(\tau, s)$ must satisfy. Note that neither this function itself, nor its derivatives are included in the record of system (1.2.7). Here we can try to use the freedom of choice to construct $Q(\tau, s)$ that is as simple as possible.

For example, let the function $P(\tau, s)$, «penalizing» violation of the condition $s \le 0$ be such that

$$\frac{\partial P}{\partial s} = \sqrt{\frac{s^2}{4\tau^2} + 1} + \frac{s}{2\tau}.$$

In this case, the equality $s = \dfrac{\partial P}{\partial s}(\tau, Q)$ will define the function $Q(\tau, s)$, which is inverse to the function $\dfrac{\partial P}{\partial s}$. That is,

$$s = \sqrt{\frac{Q^2}{4\tau^2} + 1} + \frac{Q}{2\tau}.$$

This implies that

$$Q(\tau, s) = \tau\left(s - \frac{1}{s}\right),$$

and system (1.2.7) takes the form

$$\begin{cases} -6 + \overline{x}_1 + 2\overline{x}_2 &= \tau\left(\overline{\lambda}_1 - \dfrac{1}{\overline{\lambda}_1}\right), \\ -6 + 2\overline{x}_1 + \overline{x}_2 &= \tau\left(\overline{\lambda}_2 - \dfrac{1}{\overline{\lambda}_2}\right), \\ -2 + \overline{\lambda}_1 + 2\overline{\lambda}_2 &= -\tau\left(\overline{x}_1 - \dfrac{1}{\overline{x}_1}\right), \\ -3 + 2\overline{\lambda}_1 + \overline{\lambda}_2 &= -\tau\left(\overline{x}_2 - \dfrac{1}{\overline{x}_2}\right). \end{cases} \tag{1.2.8}$$

Its solutions[4] are given in Table 1.2.1a and Table 1.2.1b for different values of the parameter $\tau$.

The calculation results show that in the primary problem the constraints $f_1(X) \le 0$ and $f_2(X) \le 0$ are active at the solution point. The constraints $g_1(\Lambda) \ge 0$ and $g_2(\Lambda) \ge 0$ are active in the dual problem.

The corresponding values of the feedback functions are numbers that are small in absolute value.

The solutions of both the primary and dual problems are unique and have finite values.

[4] In reality, these solutions were obtained from system (4.2.3).

T a b l e 1.2.1a.
Solutions of system (1.2.8) for example 1.2.1

| $\tau$ | $\overline{x}_{1\tau}$ | $\overline{x}_{2\tau}$ | $F(\overline{X}_\tau)$ | $\overline{\lambda}_{1\tau}$ | $\overline{\lambda}_{2\tau}$ | $G(\overline{\Lambda}_\tau)$ |
|---|---|---|---|---|---|---|
| $10^{-1}$ | 1.787429330 | 2.129893665 | 9.964539653 | 1.263573111 | 0.306815112 | 9.422329338 |
| $10^{-2}$ | 1.979982065 | 2.012885024 | 9.998619202 | 1.328142519 | 0.328554106 | 9.940179747 |
| $10^{-3}$ | 1.998024716 | 2.001278917 | 9.999886182 | 1.332831445 | 0.332835513 | 9.994001742 |
| $10^{-4}$ | 1.999802747 | 2.000127789 | 9.999988862 | 1.333283314 | 0.333283355 | 9.999400017 |
| $10^{-5}$ | 1.999980277 | 2.000012778 | 9.999998889 | 1.333328333 | 0.333328334 | 9.999940000 |
| $10^{-6}$ | 1.999998028 | 2.000001278 | 9.999999889 | 1.333332833 | 0.333332833 | 9.999994000 |
| $10^{-7}$ | 1.999999803 | 2.000000128 | 9.999999989 | 1.333333283 | 0.333333283 | 9.999999400 |
| $10^{-8}$ | 1.999999980 | 2.000000013 | 9.999999999 | 1.333333328 | 0.333333328 | 9.999999940 |

T a b l e 1.2.1b.
Solutions of system (1.2.8) for example 1.2.1

| $\tau$ | $f_1\big(\overline{X}\big),\ Q(\tau,\overline{\lambda}_1)$ | $f_2\big(\overline{X}\big),\ Q(\tau,\overline{\lambda}_2)$ | $g_1\big(\overline{\Lambda}\big),\ -Q(\tau,\overline{x}_1)$ | $g_2\big(\overline{\Lambda}\big),\ -Q(\tau,\overline{X}_2)$ |
|---|---|---|---|---|
| $10^{-1}$ | 0.047216659 | -0.295247676 | -0.122796665 | -0.166038666 |
| $10^{-2}$ | 0.005752113 | -0.027150846 | -0.014749270 | -0.015160857 |
| $10^{-3}$ | 0.000582549 | -0.002671652 | -0.001497530 | -0.001501598 |
| $10^{-4}$ | 0.000058326 | -0.000266717 | -0.000149975 | -0.000150016 |
| $10^{-5}$ | 0.000005833 | -0.000026667 | -0.000015000 | -0.000015000 |
| $10^{-6}$ | 0.000000583 | -0.000002667 | -0.000001500 | -0.000001500 |
| $10^{-5}$ | 0.000000058 | -0.000000267 | -0.000000150 | -0.000000150 |
| $10^{-6}$ | 0.000000006 | -0.000000027 | -0.000000015 | -0.000000015 |

For completeness of description, let us also consider another version of example 1.2.1, taking its objective function with coefficient 1 at $x_1$.

`Example 1.2.2` . For a pair of problems:

**primary problem**

*maximize in* $E^2$ *the function* $F(X) = x_1 + 3x_2,$
*subject to* $x_1 \geq 0\ ,\ x_2 \geq 0$
*and* $x_1 + 2x_2 \leq 6\ ,\ \ 2x_1 + x_2 \leq 6\ ;$

**dual problem**

*minimize in* $E^2$ *the function* $G(\Lambda) = 6\lambda_1 + 6\lambda_2$,
*subject to* $\lambda_1 \geq 0\,,\ \lambda_2 \geq 0$
*and*
$\lambda_1 + 2\lambda_2 \geq 1\,,\ \ 2\lambda_1 + \lambda_2 \geq 3\,.$

The solutions of this pair of mutually dual problems are

$$x_1^* = 0,\ x_2^* = 3,\ F^* = 9 \qquad \text{and} \qquad \lambda_1^* = \tfrac{3}{2},\ \lambda_2^* = 0,\ G^* = 9\,.$$

Graphically, the solutions are shown in Fig. 1.2.2 by points $A^*$ and $B^*$.

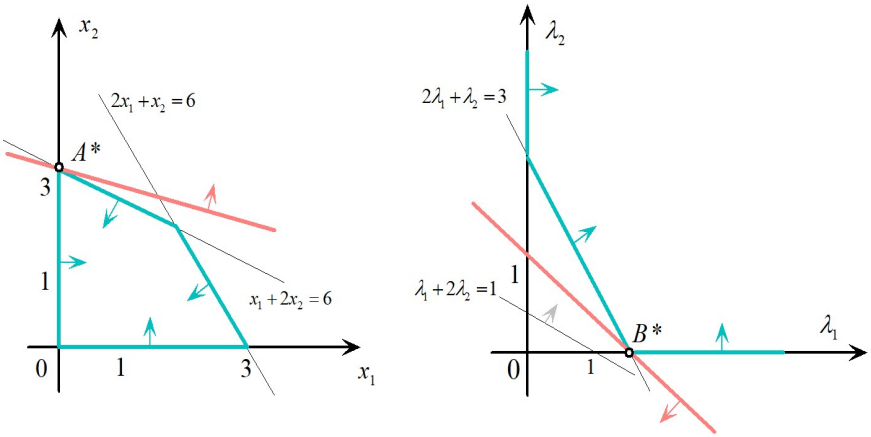



Fig. 1.2.2. Graphical representation of problem 1.2.2

System (1.2.7) in this case has the form

$$\begin{cases} -6 + \overline{x}_1 + 2\overline{x}_2 &= \tau\left(\overline{\lambda}_1 - \dfrac{1}{\overline{\lambda}_1}\right), \\ -6 + 2\overline{x}_1 + \overline{x}_2 &= \tau\left(\overline{\lambda}_2 - \dfrac{1}{\overline{\lambda}_2}\right), \\ -1 + \overline{\lambda}_1 + 2\overline{\lambda}_2 &= -\tau\left(\overline{x}_1 - \dfrac{1}{\overline{x}_1}\right), \\ -3 + 2\overline{\lambda}_1 + \overline{\lambda}_2 &= -\tau\left(\overline{x}_2 - \dfrac{1}{\overline{x}_2}\right), \end{cases} \tag{1.2.9}$$

whose solutions for various values of the parameter $\tau$ are given in Table 1.2.2a and Table 1.2.2b.

In this case, in the primary problem the constraints $f_1(X) \leq 0$ and $x_1 \geq 0$ are active at the solution point, and in the dual problem, the constraints $g_2(\Lambda) \geq 0$ and $\lambda_2 \geq 0$ are active. The values of the feedback functions corresponding to these constraints are small in absolute value numbers.

The solutions of both the primary and dual problems 1.2.2 are unique and finite.

T a b l e 1.2.2a.
Solutions of system (1.2.9) for example 1.2.2

| $\tau$ | $\overline{x}_{1\tau}$ | $\overline{x}_{2\tau}$ | $F(\overline{X}_\tau)$ | $\overline{\lambda}_{1\tau}$ | $\overline{\lambda}_{2\tau}$ | $G(\overline{\Lambda}_\tau)$ |
|---|---|---|---|---|---|---|
| $10^{-1}$ | 0.222133379 | 2.919567131 | 8.980834772 | 1.352207746 | 0.037879447 | 8.340523161 |
| $10^{-2}$ | 0.020326771 | 2.993894717 | 9.002010921 | 1.485014527 | 0.003372129 | 8.930319938 |
| $10^{-3}$ | 0.002003327 | 2.999413919 | 9.000245086 | 1.498500135 | 0.000333714 | 8.993003095 |
| $10^{-4}$ | 0.000200033 | 2.999941639 | 9.000024951 | 1.499850001 | 0.000033337 | 8.999300031 |
| $10^{-5}$ | 0.000020000 | 2.999994166 | 9.000002500 | 1.499985000 | 0.000003333 | 8.999930000 |
| $10^{-6}$ | 0.000002000 | 2.999999417 | 9.000000250 | 1.499998500 | 0.000000333 | 8.999993000 |
| $10^{-7}$ | 0.000000200 | 2.999999942 | 9.000000025 | 1.499999850 | 0.000000033 | 8.999999300 |
| $10^{-8}$ | 0.000000020 | 2.999999994 | 9.000000002 | 1.499999985 | 0.000000003 | 8.999999930 |

T a b l e 1.2.2b.
Solutions of system (1.2.9) for example 1.2.2

| $\tau$ | $f_1\left(\overline{X}\right),\ Q(\tau, \overline{\lambda}_1)$ | $f_2\left(\overline{X}\right),\ Q(\tau, \overline{\lambda}_2)$ | $g_1\left(\overline{\Lambda}\right),\ -Q(\tau, \overline{x}_1)$ | $g_2\left(\overline{\Lambda}\right),\ -Q(\tau, \overline{x}_2)$ |
|---|---|---|---|---|
| $10^{-1}$ | 0.061267641 | -2.636166110 | 0.427966641 | -0.257705060 |
| $10^{-2}$ | 0.008116204 | -2.965451741 | 0.491758786 | -0.026598816 |
| $10^{-3}$ | 0.000831166 | -2.996579426 | 0.499167563 | -0.002666015 |
| $10^{-4}$ | 0.000083312 | -2.999658294 | 0.499916676 | -0.000266660 |
| $10^{-5}$ | 0.000008333 | -2.999965833 | 0.499991667 | -0.000026667 |
| $10^{-6}$ | 0.000000833 | -2.999996583 | 0.499999167 | -0.000002667 |
| $10^{-7}$ | 0.000000083 | -2.999999658 | 0.499999917 | -0.000000267 |
| $10^{-8}$ | 0.000000008 | -2.999999966 | 0.499999992 | -0.000000027 |

## 1.3. Feedback functions in nonlinear problems

We will generalize the feedback function method to the case of nonlinear problems as follows.

Note first that the equalities

$$\begin{cases} -\beta_i + \sum\limits_{j=1}^{n} \alpha_{ij} x_j - Q(\tau, \lambda_i) &= \dfrac{\partial U}{\partial \lambda_i} \qquad i = \overline{1,m}\,, \\ -\sigma_j + \sum\limits_{i=1}^{m} \alpha_{ij} \lambda_i + Q(\tau, \overline{x}_j) &= \dfrac{\partial U}{\partial x_j} \qquad j = \overline{1,n}\,. \end{cases}$$

are true for function

$$U(\tau, X, \Lambda) = \sum_{j=1}^{n} \Big( \sigma_j x_j - R(\tau, x_j) \Big) + \sum_{i=1}^{m} \Big( \beta_i \lambda_i + R(\tau, \lambda_i) \Big) - \sum_{j=1}^{n} \sum_{i=1}^{m} \alpha_{ij} x_j \lambda_i\,, \tag{1.3.1}$$

Where $R(\tau, s) = \int\limits_{\sigma(\tau)}^{s} Q(\tau, u)\, du$, and $\sigma(\tau)$ is uniquely found from the equation $Q\big(\tau, \sigma(\tau)\big) = 0$.

Indeed, the equation $Q\big(\tau, \sigma\big) = 0$ has a unique positive root $\sigma$ for each fixed $\tau > 0$, since the function $Q(\tau, s)$ is strictly monotonically increasing in $s$ and is not bounded either below or above $\forall s \in (0, +\infty)$.

In other words, the solutions of system (1.2.7) $\overline{X}_\tau$ and $\overline{\Lambda}_\tau$ are *stationary points* of the function $U(\tau, X, \Lambda)$ in the variables $X$ and $\Lambda$. Moreover, the function (1.3.1) and the conditions for its stationarity can be written as

$$U(\tau, X, \Lambda) = L(X, \Lambda) - \sum_{j=1}^{n} R\big(\tau, x_j\big) + \sum_{i=1}^{m} R\big(\tau, \lambda_i\big)\,,$$

And

$$\begin{cases} \dfrac{\partial L}{\partial x_j} = \quad Q(\tau, x_j) \quad j = \overline{1,n}\,, \\ \dfrac{\partial L}{\partial \lambda_i} = -Q(\tau, \lambda_i) \quad i = \overline{1,m}\,, \end{cases} \iff$$

$$\iff \quad \begin{cases} \dfrac{\partial L}{\partial x_j} \quad = Q(\tau, x_j) \quad j = \overline{1,n}\,, \\ f_i(X) = Q(\tau, \lambda_i) \quad i = \overline{1,m}\,, \end{cases}$$

where

$$L(X,\Lambda)=\sum_{j=1}^{n}\sigma_j x_j-\sum_{i=1}^{m}\lambda_i\Big(-\beta_i+\sum_{j=1}^{n}\alpha_{ij}x_j\Big)=F(X)-\sum_{i=1}^{m}\lambda_i f_i(X)$$

is the regular Lagrange function of problem (1.2.1).

This form of the Lagrange function does not depend on whether the functions $F(X),\ f_i(X)\quad i=\overline{1,m}$ are linear or not. Therefore, $U(\tau,X,\Lambda)$ can be introduced *by definition* also for the nonlinear problem (1.1.1).

Suppose that the conditions $x_j\geq 0\quad j=\overline{1,n}$ are contained in the system of inequalities $f_i(X)\leq 0\quad i=\overline{1,m}$. Then the function $U(\tau,X,\Lambda)$ and its stationarity conditions for problem (1.1.1) will respectively be

$$U(\tau,X,\Lambda)=L(X,\Lambda)+\sum_{i=1}^{m}R\big(\tau,\lambda_i\big)\,, \tag{1.3.2}$$

and

$$\left\{\begin{array}{rcll}\dfrac{\partial L}{\partial x_j}\big(\overline{X},\overline{\Lambda}\big) & = & 0 & j=\overline{1,n}\,,\\ \dfrac{\partial L}{\partial \lambda_i}\big(\overline{X},\overline{\Lambda}\big) & = & -Q(\tau,\overline{\lambda}_i) & i=\overline{1,m}\,,\end{array}\right.\quad\Longleftrightarrow$$

$$\Longleftrightarrow\quad\left\{\begin{array}{rcll}\dfrac{\partial L}{\partial x_j}\big(\overline{X},\overline{\Lambda}\big) & = & 0 & j=\overline{1,n}\,,\\ f_i(\overline{X}) & = & Q(\tau,\overline{\lambda}_i) & i=\overline{1,m}\,,\end{array}\right. \tag{1.3.3}$$

Where $L(X,\Lambda)=F(X)-\sum\limits_{i=1}^{m}\lambda_i f_i(X)$.

Here, as in the linear case, we can try to use $F\big(\overline{X}\big)$ as an approximation of the value of $F^*$ for problem (1.1.1). The possible validity of these assumptions is indicated by

`Example 1.3.1:` *maximize with respect to* $X\in E^1$ *the function*

$$F(X)=\frac{1}{3}x^3-\frac{4}{3}x$$

*subject to* $\quad f(X)=x-3\leq 0\,,$

This problem has two local solutions [5]

$$x^*_{(1)}=-\frac{2}{3}\sqrt{3}\approx -1.154700538\,,\qquad F^*_{(1)}=\frac{16}{27}\sqrt{3}\approx 1.026499479$$

[5]Here and below, the subscript of the form *an integer in parentheses* is used to denote ordinal number of an element in a set, while a subscript without brackets is the number of a component of a vector or matrix.

and $\qquad X^*_{(2)} = 3, \qquad F^*_{(2)} = 7\frac{2}{3}.$

To solve the problem, we use a feedback function of the form

$$Q(\tau, s) = \tau\left(s - \frac{1}{s}\right) \qquad \text{with} \qquad R(\tau, s) = \tau\left(\frac{s^2}{2} - \ln s - \frac{1}{2}\right).$$

Their graphs for various values of $\tau$ are shown in Fig. 1.3.3.

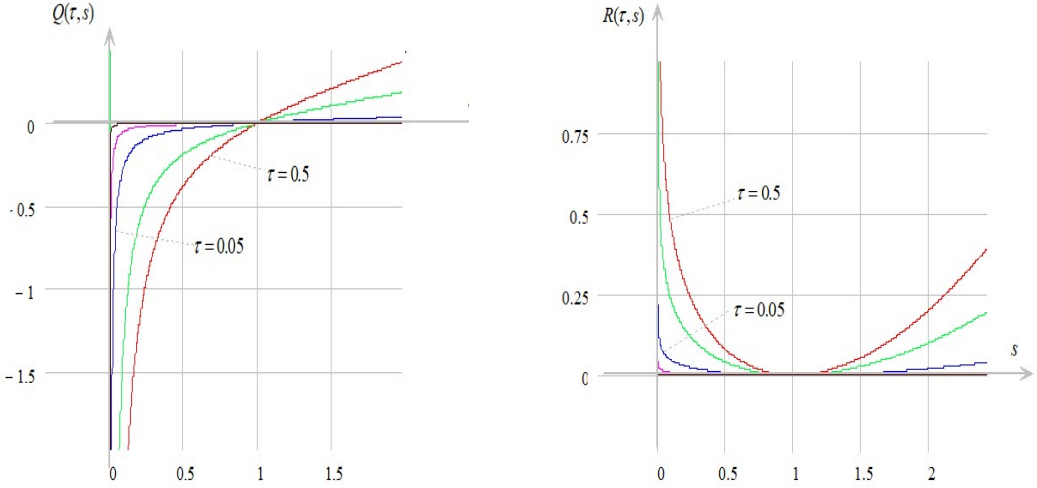



Fig. 1.3.3. Graphs of the functions $Q(\tau, s)$ and $R(\tau, s)$

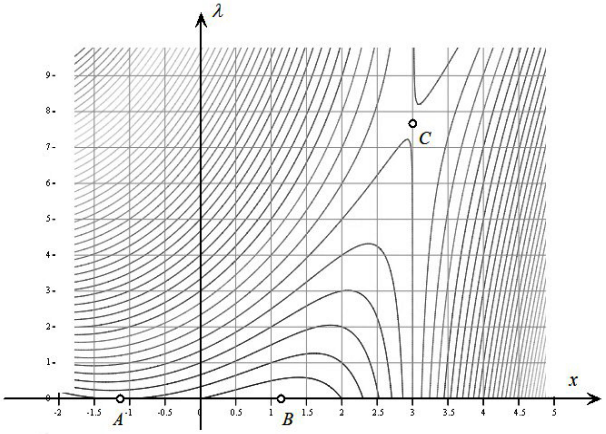



Fig. 1.3.1. System of isolines for the function $U(\,0.001,\, x,\, \lambda\,)$

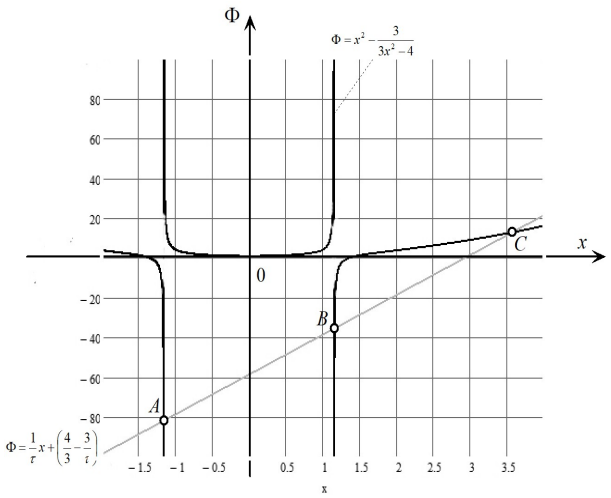



Fig. 1.3.2. Graphical solution of equation (1.3.5)

In this case, the $U$-function will be

$$U(\tau, X, \Lambda) = \frac{1}{3}x^3 - \frac{4}{3}x - \lambda(-3 + x) + \tau\left(\frac{\lambda^2}{2} - \ln\lambda - \frac{1}{2}\right), \qquad (1.3.4)$$

and the conditions of its stationarity, that is, system (1.3.2), respectively

$$\begin{cases} \overline{x}^2 - \dfrac{4}{3} - \overline{\lambda} &= 0\,, \\ 3 - \overline{x} &= -\tau\left(\overline{\lambda} - \dfrac{1}{\overline{\lambda}}\right). \end{cases}$$

The properties of the $U$-function (1.3.4) are illustrated by Fig. 1.3.1 and Fig. 1.3.2.

Fig. 1.3.1 shows a system of isolines of the function $U\Big(\tau,\, X,\, \Lambda\Big)$ for the problem from example 1.3.1 with a fixed value of $\tau = 0.001$. In Fig. 1.3.2 we get a graphical solution of the equation

$$\frac{\overline{x} - 3}{\tau} = \overline{x}^2 - \frac{4}{3} - \frac{3}{3\overline{x}^2 - 4}. \qquad (1.3.5)$$

This equation is obtained by excluding $\overline{\lambda}$ from the stationarity conditions for the function (1.3.4). Here the value of the parameter $\tau$ is equal 0.05.

Based on Fig. 1.3.1 and 1.3.2, we can assume that the auxiliary function $U\Big(\tau,\, x,\, \lambda\Big)$ for small positive values of $\tau$ has three isolated stationary points belonging, respectively, to the neighborhoods of points designated as $A, B$ and $C$ :

$$\Big\{\, x^*_{(A)} = -\frac{2}{3}\sqrt{3},\;\; \lambda^*_{(A)} = 0 \,\Big\}, \quad \Big\{\, x^*_{(B)} = \frac{2}{3}\sqrt{3},\;\; \lambda^*_{(B)} = 0 \,\Big\}$$

and

$$\Big\{\, x^*_{(C)} = 3,\;\; \lambda^*_{(C)} = 7\frac{2}{3} \Big\}.$$

Table 1.3.1 — 1.3.3 present data obtained by the same algorithm. They can be used to estimate the rate of convergence to the solution at $\tau \to +0$.

It is also worth noting that the resulting approximation is local. In addition, it depends not only on the initial approximation for $x$, but also on the value of the parameter $\tau$.

Thus, data from Table 1.3.1 — 1.3.3 can be used as approximate estimates of the solutions to problem (1.1.1).

The problem of reducing the error of the feedback function method will be considered separately in § 2.4.

T a b l e 1.3.1.
Solution of system 1.3.3 with initial point $x = 2$ and $\lambda = 5$.

| $\tau$ | $\overline{X}(\tau)$ | $\overline{\lambda}(\tau)$ | $F(\overline{x}(\tau))$ | $L(\overline{x}(\tau), \overline{\lambda}(\tau))$ | $U(\tau, \overline{x}(\tau), \overline{\lambda}(\tau))$ |
|---|---|---|---|---|---|
| $10^{-1}$ | 1.178158150 | 0.054725148 | -1.025760791 | -0.926060226 | -0.685367291 |
| $10^{-2}$ | 3.080324420 | 8.155065201 | 5.635349344 | 4.980298460 | 5.286837510 |
| $10^{-3}$ | 3.007582555 | 7.712219493 | 5.058305553 | 4.999827224 | 5.027023583 |
| $10^{-4}$ | 3.000754080 | 7.671191736 | 5.005783012 | 4.999998294 | 5.002686906 |
| $10^{-5}$ | 3.000075367 | 7.667118874 | 5.000577830 | 4.999999983 | 5.000268537 |
| $10^{-6}$ | 3.000007536 | 7.666711884 | 5.000057778 | 5.000000000 | 5.000026852 |
| $10^{-7}$ | 3.000000754 | 7.666671188 | 5.000005778 | 5.000000000 | 5.000002685 |
| $10^{-8}$ | 3.000000075 | 7.666667119 | 5.000000578 | 5.000000000 | 5.000000269 |

T a b l e 1.3.2.
Solution of system 1.3.3 with initial point $x = -2$ and $\lambda = 0.00001$.

| $\tau$ | $\overline{X}(\tau)$ | $\overline{\Lambda}(\tau)$ | $F(\overline{X}(\tau))$ | $L(\overline{X}(\tau), \overline{\Lambda}(\tau))$ | $U(\tau, \overline{X}(\tau), \overline{\Lambda}(\tau))$ |
|---|---|---|---|---|---|
| $10^{-1}$ | -1.165044573 | 0.023995523 | 1.026276558 | 1.126218979 | 1.449236571 |
| $10^{-2}$ | -1.155742025 | 0.002406295 | 1.026399226 | 1.036399168 | 1.091695866 |
| $10^{-3}$ | -1.154804753 | 0.000240685 | 1.026400466 | 1.027400466 | 1.035232487 |
| $10^{-4}$ | -1.154710961 | 0.000024069 | 1.026400478 | 1.026500478 | 1.027513937 |
| $10^{-5}$ | 1.154702885 | 0.000005419 | -1.026400479 | -1.026390479 | -1.026274223 |
| $10^{-6}$ | 1.154700773 | 0.000000542 | -1.026400479 | -1.026399479 | -1.026385550 |
| $10^{-7}$ | 3.000000754 | 7.666671188 | 5.000005778 | 5.000000000 | 5.000002685 |
| $10^{-8}$ | 3.000000075 | 7.666667119 | 5.000000578 | 5.000000000 | 5.000000269 |

T a b l e 1.3.3.
Solution of system 1.3.3 with initial point $X = 2$ and $\lambda = 0.00001$.

| $\tau$ | $\overline{X}(\tau)$ | $\overline{\Lambda}(\tau)$ | $F(\overline{X}(\tau))$ | $L(\overline{X}(\tau), \overline{\Lambda}(\tau))$ | $U(\tau, \overline{X}(\tau), \overline{\Lambda}(\tau))$ |
|---|---|---|---|---|---|
| $10^{-1}$ | 1.178158936 | 0.054725145 | -1.025760748 | -0.926060232 | -0.685367291 |
| $10^{-2}$ | 1.157047644 | 0.005425916 | -1.026394113 | -1.016394408 | -0.969228576 |
| $10^{-3}$ | 1.154935201 | 0.000541986 | -1.026400415 | -1.025400415 | -1.018380145 |
| $10^{-4}$ | 1.154724004 | 0.000054192 | -1.026400478 | -1.026300478 | -1.025368181 |
| $10^{-5}$ | 1.154702885 | 0.000005419 | -1.026400479 | -1.026390479 | -1.026274223 |
| $10^{-6}$ | 1.154700773 | 0.000000542 | -1.026400479 | -1.026399479 | -1.026385550 |
| $10^{-7}$ | 3.000000754 | 7.666671188 | 5.000005778 | 5.000000000 | 5.000002685 |
| $10^{-8}$ | 3.000000075 | 7.666667119 | 5.000000578 | 5.000000000 | 5.000000269 |

# Chapter 2.

# Feedback Function Properties

## 2.1. Convergence of the Feedback Function Method

To justify the feedback function method it is necessary to show that the equality

$$\lim_{\tau \to +0} U\left(\tau, \overline{X}_\tau, \overline{\Lambda}_\tau\right) = F(X^*)$$

is valid for each solution to the problem (1.1.1).

We first note that the system of conditions (1.3.3) of stationarity of the auxiliary function $U\left(\tau, X, \Lambda\right)$ does not contain constraints of the «inequality» type. But its structure is similar to the formulation of Theorem 1.1.1 (Karush-Kuhn-Tucker). Therefore, the function $U\left(\tau, X, \Lambda\right)$ can be considered as a *modification* of the Lagrange function.

This modification consists of adding to the regular Lagrange function terms with $R(\tau, s)$. For this reason, we will first study the properties of this supplement.

We will assume that *nonlinear* problem (1.1.1) has a local solution with a finite value $F^*$ at, perhaps, not a single point $X^*$.

Let for this problem the regular Lagrange function

$$L(X, \Lambda) = F(X) - \sum_{i=1}^{m} \lambda_i f_i(X)\,,$$

have a stationary point $\{X^*;\Lambda^*\}$ with $L(X^*,\Lambda^*)=F^*$.

We also assume that there exist compact sets with nonempty interior $\Omega_x \subseteq E^n \quad \Omega_\lambda \subseteq E_+^m$ [1] such that $X^* \in \operatorname{int}\Omega_x$ and $\Lambda^* \in \operatorname{int}\Omega_\lambda$. Moreover, the function $L(X,\Lambda)$ under the assumptions made is continuous in $\Omega=\Omega_x\otimes\Omega_\lambda$.

Now we will introduce the *feedback function* $Q(\tau,s) \quad \forall\tau,s>0$.

Let us choose its form using condition (1.1.4). It follows from this condition that the function $\dfrac{\partial P}{\partial s}(\tau,s)$ has the form $\Phi\left(\dfrac{s}{\tau}\right)$. In this case, $\Phi(u)$ is a strictly monotone and continuously differentiable function of *one* argument $u=\dfrac{s}{\tau}$.

Then $Q(\tau,s)=\tau\Psi(s)$, where $\Psi(s)$ is a function *inverse* to the function $\Phi(u)$. Indeed, by the definition of the inverse function:

$$\frac{\partial P}{\partial s}(\tau,Q)=s \qquad \Longrightarrow \qquad \Phi\left(\frac{Q}{\tau}\right)=s \qquad \Longrightarrow$$

$$\Longrightarrow \qquad \frac{Q}{\tau}=\Psi(s) \qquad \Longrightarrow \qquad Q(\tau,s)=\tau\Psi(s)\,.$$

By virtue of the assumptions made, the function $\Psi(s)$ has the following properties:

1°. $\Psi(s)$ is strictly monotonically increasing in $s$ and at the same time
$$\lim_{s\to+0}\Psi(s)=-\infty, \quad \lim_{s\to+\infty}\Psi(s)=+\infty\,.$$

2°. In its domain, the function $\Psi(s)$ is continuously differentiable.

Moreover, $\forall s\in(0,+\infty)$ there exists a non-negative function $R(\tau,s)$ such that

$$R(\tau,s)=\tau\int_\theta^s \Psi(u)\,du\,, \tag{2.1.1}$$

where $\theta$ is a solution of the equation $\Psi\big(\theta\big)=0$. It follows from property 1° that the solution exists and is unique.

[1] $E_+^m$ denotes the positive orthant in $E^m$.

Generalizing (1.3.2), we introduce *by definition* for problem (1.1.1) an auxiliary function

$$U(\tau, x, \Lambda) \;=\; L(x, \Lambda) + W(\tau, \Lambda)\,, \qquad \text{where} \quad W(\tau, \Lambda) \;=\; \sum_{i=1}^{m} R(\tau, \lambda_i)\,. \tag{2.1.2}$$

The properties of the functions $R(\tau, s)$, $W(\tau, \Lambda)$ and $U(\tau, X, \Lambda)$ are described by the following statements.

`Lemma 2.1.1.` **The function $R(\tau, s)$ :**

**1) is non-negative, twice continuously differentiable and the limit equality**

$$\lim_{\tau \to +0} R(\tau, s) = 0 \qquad \forall s \in (0, +\infty)\,,$$

**and this limit transition is monotone.**

**2) is strictly convex for $s > 0$ and $\dfrac{\partial^2 R}{\partial\, s^2} > 0\,,$**

**3) has at $s = \theta$ a unique minimum, at which $R\big(\tau, \theta\big) = 0\,.$**

`Proof.`

1°. Monotonicity of the limit transition $\lim\limits_{\tau \to +0} R(\tau, s) = 0$ follows from (2.1.1) , and from the definitions of $\theta$ and the function $\Psi(s)$ it follows that

$$\Psi(s) > 0 \;\text{ for }\; s > \theta \qquad \text{and} \qquad \Psi(s) < 0 \;\text{ for }\; s < \theta\,.$$

Then, by the properties of the integral (2.1.1), the function $R(\tau, s)$ is positive for $s \neq \theta$ and is equal to 0 for $s = \theta$. In this case, from equality (2.1.1) follows the validity of the limit relations specified in item 1) of the lemma.

2°. We have

$$Q(\tau, s) = \frac{\partial R}{\partial\, s} \qquad \text{and} \qquad \frac{\partial Q}{\partial\, s} = \frac{\partial^2 R}{\partial\, s^2} \qquad \forall s > 0.$$

Due to the strict monotonicity and continuous differentiability of the function $\Psi(s)$ with respect to $s$ it follows that the function $R(\tau, s)$ is twice continuously differentiable. It also has a

positive second derivative with respect to the argument $s$. This means that the function $R(\tau, s)$ strictly convex in $s$ for $s > 0$.

3°. Thus we have

$$R\Big(\tau, \theta\Big) = 0\,, \qquad \frac{\partial R}{\partial s}\Big(\tau, \theta\Big) = 0\,, \qquad \frac{\partial^2 R}{\partial s^2}\Big(\tau, \theta\Big) > 0\,.$$

This means that the twice continuously differentiable function $R(\tau, s)$ has a unique zero minimum at $s = \theta$.

The lemma is proved.

Lemma 2.1.2. **The function $W(\tau, \Lambda)$ has for any fixed $\tau > 0$ a unique zero minimum in $\Lambda \in E_+^m$ at the point with coordinates $\lambda_i = \theta \;\; i = \overline{1, m}$.**

Proof.

We use Lemma 2.1.1 with

1) separability of the function $W(\tau, \Lambda)$ by components $\Lambda$,

2) equalities $\dfrac{\partial W}{\partial \lambda_i} = \dfrac{\partial R}{\partial s}\Big|_{s=\theta} = Q(\tau, \theta) = 0 \qquad i = \overline{1, m}$ and

3) relations $\dfrac{\partial^2 W}{\partial \lambda_i^2} = \dfrac{\partial^2 R}{\partial s^2}\Big|_{s=\theta} > 0$ are valid $\forall i = \overline{1, m}\,, \;\; \forall \tau, s > 0.$

Then, from the sufficient condition for a strict minimum of a function of several variables it follows that the non-negative function $W(\tau, \Lambda)$ has a unique zero minimum at the point $\lambda_i = \theta \; i = \overline{1, m}$ with respect to $\Lambda \in \operatorname{int} E_+^m$.

The lemma is proved.

By construction, the set $\Omega = \Omega_x \otimes \Omega_\lambda$ is compact in $E^n \otimes E_+^m$. Then the following holds

Theorem 2.1.1. **$U$-function (2.1.2) has a local unique stationary point $\{\overline{X}(\tau); \overline{\Lambda}(\tau)\} \in \Omega$, where the vector functions $\overline{X}(\tau)$ and $\overline{\Lambda}(\tau)$ are the solution of the**

**system of equations**

$$\begin{cases} \operatorname*{grad}_x U(\tau, \overline{X}, \overline{\Lambda}) = o, \\ \operatorname*{grad}_\lambda U(\tau, \overline{X}, \overline{\Lambda}) = o \end{cases} \tag{2.1.3}$$

**for fixed** $\tau > 0$**.**

Proof.

1°. First we need to show that the function $U(\tau, X, \Lambda)$ has in the interior $\Omega$ a unique minimum in $\Lambda$ for a fixed $X \in \Omega_x$ and a unique maximum in $X$ for a fixed $\Lambda \in \Omega_\lambda$.

Let us consider the first of these statements. By the assumptions made, for $X \in \Omega_x$ $L(X, \Lambda)$ has a minimum in $\Lambda$ on $\Omega_\lambda$, and the term $W(\tau, \Lambda)$ does not depend on $X$.

The function $U(\tau, X, \Lambda)$ is continuous and bounded below on the compact set $\Omega_\lambda$ by the sum of the minima $L(X, \Lambda)$ and $W(\tau, \Lambda)$. Then it has a minimum with respect to $\Lambda \in \Omega_\lambda$. In this case, the point of this minimum must belong to $\operatorname{int}\Omega_\lambda$, since $\lim\limits_{s \to +0} R(\tau, s) = +\infty$.

If $\overline{\Lambda} = \operatorname*{argmin}\limits_{\lambda \in \operatorname{int}\Omega_\lambda} U(\tau, x, \lambda)$, then by virtue of the assumption of continuous differentiability $U$ with respect to $x$, the equality $\operatorname*{grad}_\lambda U(\tau, x, \overline{\lambda}) = o$ is valid for a fixed $x \in \operatorname{int}\Omega_x$.

2°. Reasoning as in the previous paragraph, we conclude, that the function $U(\tau, X, \Lambda)$ has a maximum with respect to $X$ on the interior of $\Omega_x$.

If $\overline{x} = \operatorname*{argmax}\limits_{x \in \operatorname{int}\Omega_x} U(\tau, X, \Lambda)$, then by virtue of the assumption of continuous differentiability $U$ with respect to $X$, the equality $\operatorname*{grad}_x U(\tau, \overline{X}, \Lambda) = o$ will also hold for any fixed $\Lambda \in \operatorname{int}\Omega_\lambda$.

3°. Finally, based on Theorem XX (*on a stationary point*) ( [9], ch. 3, §16 ) , from which in the notations used it follows:

> *if* $\Omega_x$ *and* $\Omega_\lambda$ *are compact, and* $U(\tau, X, \lambda)$ *is continuous, has a strict maximum with respect to* $X$ *on* $\operatorname{int}\Omega_x$ $\forall \Lambda \in \operatorname{int}\Omega_\lambda$ *and has a strict minimum with respect to* $\Lambda$ *on* $\operatorname{int}\Omega_\lambda$ $\forall X \in \operatorname{int}\Omega_x$, *then the function* $U(\tau, X, \Lambda)$ *on the interior of the compact* $\Omega = \Omega_x \otimes \Omega_\lambda$ *has a unique stationary point, defined by the system of equations* (2.1.3).

From which we come to the conclusion about the validity of the statement of the theorem.

`The theorem is proved.`

The system of equations (2.1.3) can also be written as

$$\begin{cases} \dfrac{\partial L}{\partial x_j} = o & j = \overline{1,n}\,, \\ \dfrac{\partial L}{\partial \lambda_i} = -Q(\tau, \lambda_i) & i = \overline{1,m} \end{cases}$$

or

$$\begin{cases} \dfrac{\partial F}{\partial x_j}(\overline{X}) - \sum\limits_{i=1}^{m} \overline{\lambda}_i \dfrac{\partial f_i}{\partial x_j}(\overline{X}) = o & j = \overline{1,n}\,, \\ f_i(\overline{X}) = -Q(\tau, \overline{\lambda}_i) & i = \overline{1,m}\,, \end{cases} \tag{2.1.4}$$

which justifies the use of the name «*feedback function*» for the function $Q(\tau, s)$.

Indeed, let us look at the second group of constraints of the system (2.1.4). Here, the active constraints of problem (1.1.1) correspond to small absolute values of $Q(\tau, \overline{\lambda}_i)$. For inactive constraints, the situation is the opposite.

In addition, the conditions of *complementary slackness* must be satisfied from $\overline{\lambda}_i f_i(\overline{X}) = -\overline{\lambda}_i Q(\tau, \overline{\lambda}_i) = -\overline{\lambda}_i \tau \Psi(\overline{\lambda}_i) \to 0$ when $\tau \to +0$.

Further, the vector function $\{\overline{X}(\tau); \overline{\Lambda}(\tau)\}$ will be called the *stationary trajectory* of the function $U(\tau, X, \Lambda)$. We will also assume that $\exists \tau_0 > 0$, such that $\forall \tau \in (0, \tau_0]$

$$\begin{cases} \overline{X}(\tau) \in \Omega_x, \\ \overline{\Lambda}(\tau) \in \Omega_\lambda. \end{cases}$$

`Theorem 2.1.2.` *On the stationary trajectory of problem* (1.1.1)

$$\lim_{\tau \to +0} U\Big(\tau, \overline{X}(\tau), \overline{\Lambda}(\tau)\Big) = F^*, \tag{2.1.5}$$

*and in the case of local uniqueness of the solution to problem* (1.1.1) *the equalities are also true*

$$\lim_{\tau \to +0} \overline{X}(\tau) = X^* \qquad \textit{and} \qquad \lim_{\tau \to +0} \overline{\Lambda}(\tau) = \Lambda^*\,. \tag{2.1.6}$$

Proof.

1°. Let $\tau \in (0, \tau_0]$. Let us estimate the difference between the values of the auxiliary function $U$ and the Lagrange function $L$ on the set $\Omega$.
Consider a fixed vector $\{X; \Lambda\} \in \Omega$. For this vector, by (2.1.2), the following inequalities hold:

$$U\Big(\tau, X, \Lambda\Big) \quad = \quad L\Big(X, \Lambda\Big) + \sum_{i=1}^{m} R\Big(\tau, \lambda_i\Big) \quad \geq \quad L\Big(X, \Lambda\Big)$$

On the other hand, since $\Omega$ is compact,

$$U\Big(\tau, X, \Lambda\Big) \quad = \quad L\Big(X, \Lambda\Big) + \sum_{i=1}^{m} R\Big(\tau, \lambda_i\Big) \quad \leq$$

$$\leq \quad L\Big(X, \Lambda\Big) + m \sup_{\left\{\begin{array}{c}\{x\,\lambda\} \in \Omega \\ i = \overline{1,m}\end{array}\right\}} R\Big(\tau, \lambda_i\Big).$$

2°. On the set $\Omega$ by virtue of the assumptions made the limiting operations are monotone, which makes it possible to apply lemma 1.1.1 (or the more general lemma 5.1 [10], ch. 6, § 5 ), and therefore they commute with the optimization operations.
Then by point 1) of Lemma 1.1.1 we have $i = \overline{1,m}$

$$\lim_{\tau \to +0} \sup_{\left\{\begin{array}{c}\{x\,\lambda\} \in \Omega \\ i = \overline{1,m}\end{array}\right\}} R\Big(\tau, \lambda_i\Big) \quad = \quad \sup_{\left\{\begin{array}{c}\{x\,\lambda\} \in \Omega \\ i = \overline{1,m}\end{array}\right\}} \lim_{\tau \to +0} R\Big(\tau, \lambda_i\Big) = 0$$

Thus we come to the conclusion that

$$\lim_{\tau \to +0} U(\tau, X, \Lambda) = L(X, \Lambda) \qquad \forall \{X; \Lambda\} \in \Omega.$$

Then in $\Omega$ for any trajectory such that $\left\| \begin{array}{c} X(t) \\ \Lambda(t) \end{array} \right\| \to \left\| \begin{array}{c} X^* \\ \Lambda^* \end{array} \right\|$ at $t \to +0$, due to the continuity of the Lagrange function we obtain

$$\lim_{t \to +0} U\Big(t, X(t), \Lambda(t)\Big) \quad = \quad \lim_{t \to +0} L\Big(X(t), \Lambda(t)\Big) = F^*.$$

These conditions will also be true on a stationary trajectory, therefore $\lim\limits_{\tau \to +0} U\Big(\tau, \overline{X}(\tau), \overline{\Lambda}(\tau)\Big) = F^*$.

Finally, in the case when the Lagrange function of problem (1.1.1) has an isolated stationary point $\{X^*, \Lambda^*\}$, equalities (2.1.6) will also be true.

`The theorem is proved.`

Note that Theorem 2.1.2, in combination with the properties of the function $R(\tau, s)$, allows us to use conditions both necessary and sufficient. These conditions are applicable to the regular Lagrange function, and to the function (2.1.2).

Important property of vector functions $\{\overline{X}(\tau), \overline{\Lambda}(\tau)\}$ describes

`Theorem 2.1.3.` **Vector functions $\{\overline{X}(\tau); \overline{\Lambda}(\tau)\}$ are continuously differentiable** $\forall \tau \in (0, \tau_0)$.

`Proof.`

1°. According to the assumptions made, the function $L(X, \Lambda)$ is concave in $X$ and convex in $\Lambda$ in a small neighborhood of the stationary point.
At the same time, the function $W(\tau, \Lambda)$ by Lemma 2.1.1 is strictly convex in $\Lambda$. Then, by (2.1.2), the function $U(\tau, X, \Lambda)$ is strictly concave in $X$ and strictly convex in $\Lambda$. Also the function is twice continuously differentiable in all its arguments.

For each fixed $\tau \in (0, \tau_0)$ the function $U(\tau, X, \Lambda)$ has a unique stationary point, defined by the system of equations (2.1.3). Let us verify that the theorem on *implicitly defined functions* [6] can be applied to this system.

The condition for applicability of the implicit function theorem is the *non-degeneracy* of the Hessian matrix for

$$U(\tau, X, \Lambda) = F(X) - \sum_{i=1}^{m} \lambda_i f_i(X) + \sum_{i=1}^{m} R(\tau, \lambda_i)$$

at the stationary point $\{\overline{X}(\tau), \overline{\Lambda}(\tau)\}$.
We will check this by introducing the notation:

$$\|B\| = \left\| \frac{\partial^2 U}{\partial x_j \partial x_k} \right\| \quad j = \overline{1, n}, \quad k = \overline{1, n},$$

$$\|A\| = \left\| \frac{\partial^2 U}{\partial x_j \partial \lambda_i} \right\| \quad j = \overline{1, n}, \quad i = \overline{1, m},$$

$$\|Y\| = \left\| \frac{\partial^2 U}{\partial \lambda_i \partial \lambda_t} \right\| \quad i = \overline{1,m}\,, \quad t = \overline{1,m}\,,$$

Then the Hessian matrix of the auxiliary function $U(\tau, X, \Lambda)$ in block form can be written as

$$\|H\| = \left\| \begin{array}{cc} \|B\| & \|A\| \\ \|A\|^{\mathrm{T}} & \|Y\| \end{array} \right\| .$$

Let us find the sign of $\det \|H\|$. By the property of the Schur complement (see, for example, Chapter 4, § 6, item 5 in [11]) and by virtue of the rules for operating with matrices, in the case under consideration the equality holds

$$\det \|H\| = \det \|B\| \cdot \det\Big( \|Y\| + \|A\|^{\mathrm{T}} \|B\|^{-1} \|A\| \Big) .$$

The function $U(\tau, X, \Lambda)$ is strictly concave in the components of $X$, therefore $\det \|B\| \neq 0$. On the other hand, it is strictly convex in $\Lambda$, due to which $\det \|Y\| > 0$.

Now we find the sign of the factor $\det\Big( \|Y\| + \|A\|^{\mathrm{T}} \|B\|^{-1} \|A\| \Big)$. Note that the matrix $\det \|Y\|$ can be considered as the matrix of some positive definite quadratic form in the space $E^m$. In this case, the product of matrices $\|A\|^{\mathrm{T}} \|B\|^{-1} \|A\|$ (for any value of the rank of the matrix $\|A\|$) by the corollary of the Binet–Cauchy theorem (see Chapter 4, § 5, item 6 in [11]) defines either a positive definite or a positive semidefinite quadratic form in the same space.

Then it is clear that the matrix $\|Y\| + \|A\|^{\mathrm{T}} \|B\|^{-1} \|A\|$ also defines a positive definite quadratic form in $E^m$ and has (by virtue of the Sylvester criterion) a positive determinant. Therefore, we find that

$$\det\Big( \|Y\| + \|A\|^{\mathrm{T}} \|B\|^{-1} \|A\| \Big) > 0$$

and that finally $\det \|H\| > 0$.

2°. We have the fact of local non-degeneracy of the Hessian matrix for stationary points of the auxiliary function $U(\tau, X, \Lambda)$. This, in combination with the assumptions made about the smoothness of the function $Q(\tau, s)$, allows the application of the statements of the theorem on implicit functions [6] to system (2.1.3). From which follows continuous differentiability of the vector functions $\overline{X}(\tau)$ and $\overline{\Lambda}(\tau)$.

In this case, the components of the vector functions $\dfrac{d\,\overline{X}}{d\,\tau}$ and $\dfrac{d\,\overline{\Lambda}}{d\,\tau}$ are determined $\forall\tau \in (0;\, \tau_0)$ by a system of linear equations

$$\left\| \begin{array}{cc} -\|B\| & -\|A\,\|^{\mathrm{T}} \\ -\|A\,\| & \|\,Y\| \end{array} \right\| \left\| \begin{array}{c} \left\| \dfrac{d\overline{X}}{d\,\tau} \right\| \\ \left\| \dfrac{d\,\overline{\Lambda}}{d\,\tau} \right\| \end{array} \right\| = - \left\| \begin{array}{c} \left\| \dfrac{\partial^2 U}{\partial X \partial \tau} \right\| \\ \left\| \dfrac{\partial^2 U}{\partial \Lambda\, \partial \tau} \right\| \end{array} \right\|,$$

where $\left\| \dfrac{\partial^2 U}{\partial X \partial \tau} \right\|_j = -\dfrac{\partial Q}{\partial \tau}\Big(\tau, \overline{x}_j(\tau)\Big)$, $\left\| \dfrac{\partial^2 U}{\partial \Lambda \partial \tau} \right\|_i = \dfrac{\partial Q}{\partial \tau}\Big(\tau, \overline{\lambda}_i(\tau)\Big)$,

$\left\| \dfrac{d\overline{X}}{d\tau} \right\|_j = \dfrac{d\overline{x}_j}{d\tau}$ and $\left\| \dfrac{d\overline{\Lambda}}{d\tau} \right\|_i = \dfrac{d\overline{\lambda}_i}{d\tau}$ at $j = \overline{1,n},\ i = \overline{1,m}$.

`The theorem is proved.`

## 2.2. Feedback functions in the linear case

The proof of the convergence of the feedback function method is based on the assumption of the existence for each point of $X^*$ *a nonempty compact neighborhood* in which there exists a regular and continuous Lagrange function of problem (1.1.1).

In this neighborhood the approximations of the solutions of the nonlinear problem obtained by the feedback function method turn out to be existing, continuous and sufficiently smooth vector functions.

This condition is significantly softer in the linear case.[2] We will show that approximations of solutions for *any* linear problem (1.1.1) have the same properties in *the entire* space $E^n \otimes E^m$.

We apply the method of feedback functions to solve the pair of linear problems (1.2.1)—(1.2.2).

That is, let there exist a continuously differentiable function $Q(\tau, s)$, which $\forall\tau > 0$ monotonically increases in $s$ $\forall s > 0$. Besides,

$$\lim_{s \to +0} Q(\tau, s) = -\infty \ \ \forall\tau > 0\,, \qquad \lim_{s \to +\infty} Q(\tau, s) = +\infty \ \ \forall\tau > 0$$

[2] Although system (1.2.7) is still nonlinear.

$$\text{and} \quad \lim_{\tau \to +0} Q(\tau, s) = 0 \quad \forall s > 0 \,.$$

We introduce function $R(\tau, s)$ , for which the equality

$$\frac{\partial R}{\partial s} = Q(\tau, s) \,. \tag{2.2.1}$$

is valid. This function is also *non-negative* and has a single zero $\forall s > 0$. Under the assumptions made, the function $R(\tau, s)$ exists and is unique for each specific $Q(\tau, s)$.

Let us introduce an auxiliary function of the form

$$U(\tau, X, \Lambda) = \sum_{j=1}^{n} \Big( \sigma_j x_j - R(\tau, x_j) \Big) + \sum_{i=1}^{m} \Big( \beta_i \lambda_i + R(\tau, \lambda_i) \Big) - \sum_{j=1}^{n} \sum_{i=1}^{m} \alpha_{ij} x_j \lambda_i \tag{2.2.2}$$

for a pair of problems (1.2.1) — (1.2.2).

Note that the stationary points of such a $U$-function (with respect to the set of components of the vectors $\{ X; \Lambda \}$) are the solutions of system (1.2.7).

`Lemma 2.2.1.` **The function $U(\tau, X, \Lambda)$ is strictly convex in the components of $X$ and strictly concave in the components of $\Lambda$ $\forall \tau > 0$ at any finite point with positive coordinates of the space $E^n \otimes E^m$.**

`Proof.`

From relations (2.2.1) and (2.2.2) we find that the components of the Hessian matrix for $U(\tau, X, \Lambda)$ (as functions of $X$ for fixed $\tau$ and $\Lambda$) have the form $\dfrac{\partial^2 U}{\partial x_j \partial x_k} = \dfrac{\partial Q}{\partial x_j} \cdot \delta_{jk} \quad j, k = \overline{1, n}$ (where $\delta_{jk}$ — Kronecker delta).

Then we have

$$\det \begin{Vmatrix} \dfrac{\partial^2 U}{\partial x_1^2} & \cdots & \dfrac{\partial^2 U}{\partial x_1 \partial x_k} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial^2 U}{\partial x_k \partial x_1} & \cdots & \dfrac{\partial^2 U}{\partial x_k^2} \end{Vmatrix} = \prod_{j=1}^{k} \left( -\frac{\partial Q}{\partial x_j} \right) \qquad k = \overline{1, n} \,.$$

The numbers $\dfrac{\partial Q}{\partial x_j}$ are positive, since by virtue of the relationship between the derivatives of mutually inverse functions and (1.1.3)

$$\frac{\partial Q}{\partial s} = \left(\frac{\partial^2 P}{\partial s^2}\right)^{-1} > 0\,. \tag{2.2.3}$$

Thus, in the Hessian submatrix under consideration, all principal minors of odd order are negative and of even order are positive. Therefore, according to Sylvester's criterion, the Hessian matrix of the function $U(\tau, X, \Lambda)$ is negative definite in the components of $X$. Therefore, the function $U(\tau, X, \Lambda)$ is strictly concave in $X$.

Arguing similarly (2.2.3), for the principal minors of the Hessian submatrix of the function $U(\tau, X, \Lambda)$ by the components $\Lambda$ we obtain

$$\prod_{i=1}^{k} \frac{\partial Q}{\partial \lambda_i} > 0 \qquad k = \overline{1, m}\,.$$

Then, according to Sylvester's criterion, the Hessian submatrix of the function $U(\tau, X, \Lambda)$ in components $\Lambda$ is positive definite, and the function $U(\tau, X, \Lambda)$ in components $\Lambda$ is strictly convex for any fixed $\tau$ and $X$.

`The lemma is proved.`

Now we will prove [3]

`Theorem 2.2.2.` **The system of equations (1.2.7) has a unique solution with positive components for any fixed $\tau > 0$ for *any* pair of problems (1.2.1) — (1.2.2).**

`Proof.`

1°. For each vector $\Lambda$ with finite positive components, there exists a unique finite vector $\widehat{X}(\Lambda)$ with positive components such that

$$\operatorname*{grad}_x U\big(\tau, \widehat{X}(\Lambda), \Lambda\big) = o.$$

By (2.2.2) this equality has the form

[3]The proof is bulky in form, but its idea is simple: the conclusion of the theorem follows from the continuity and monotonicity of the function $Q(\tau, s)\ \forall s > 0$.

$$Q(\tau, \widehat{x}_j(\Lambda)) - \sigma_j + \sum_{i=1}^{m} \alpha_{ij}\lambda_i = 0 \qquad j = \overline{1,n}\,. \tag{2.2.4}$$

Conditions (2.2.4) are equivalent to the equalities

$$\widehat{x}_j(\Lambda) = \frac{\partial P}{\partial s}\Big(\tau, -g_j(\Lambda)\Big) \qquad j = \overline{1,n}\,, \quad \forall \Lambda > o\,,$$

in which the right-hand sides exist, are positive, and are uniquely defined in $\Lambda$. The function $U(\tau, X, \Lambda)$ is strictly concave in $X$. As a result, this means that $\widehat{X}(\Lambda) = \operatorname*{argmax}_x U(\tau, X, \Lambda)\,.$

Arguing similarly, we conclude that for each vector $X$ with positive components there will exist a vector $\widehat{\Lambda}(X)$ with positive components such that,

$$Q(\tau, \widehat{\lambda}_i(X)) + \beta_i - \sum_{j=1}^{n} \alpha_{ij}x_j = 0 \qquad i = \overline{1,m}\,. \tag{2.2.5}$$

and that $\widehat{\Lambda}(X) = \operatorname*{argmin}_\lambda U(\tau, X, \Lambda)$ by virtue of the strict convexity of $U(\tau, X, \Lambda)$ with respect to $\Lambda$.

2°. Let us find *point* of minimum $U(\tau, \widehat{X}(\Lambda), \Lambda)$ with respect to $\Lambda$. From (2.2.2) we have

$$U(\tau, \widehat{X}(\Lambda), \Lambda) = \sum_{j=1}^{n}\Big(\sigma_j\widehat{x}_j(\Lambda) - R(\tau, \widehat{x}_j(\Lambda))\Big) + \\ + \sum_{i=1}^{m}\Big(\beta_i\lambda_i + R(\tau, \lambda_i)\Big) - \sum_{j=1}^{n}\sum_{i=1}^{m}\alpha_{ij}\widehat{x}_j(\Lambda)\lambda_i\,.$$

Due to the strict convexity of this function, a sufficient condition for its minimum will be $q = \overline{1,m}$

$$\frac{\partial U}{\partial \lambda_q}\Big(\tau, \widehat{X}(\Lambda), \Lambda\Big) = \sum_{j=1}^{n}\left[\sigma_j\frac{\partial \widehat{x}_j(\Lambda)}{\partial \lambda_q} - \frac{\partial R}{\partial s}(\tau, \widehat{x}_j(\Lambda))\,\frac{\partial \widehat{x}_j(\Lambda)}{\partial \lambda_q}\right] + \\ + \beta_q + \frac{\partial R}{\partial s}(\tau, \lambda_q) - \sum_{j=1}^{n}\alpha_{qj}\widehat{x}_j(\Lambda) - \sum_{j=1}^{n}\sum_{i=1}^{m}\alpha_{qj}\lambda_i\frac{\partial \widehat{x}_j(\Lambda)}{\partial \lambda_q} = 0$$

or, after rearranging the terms,

$$\sum_{j=1}^{n}\frac{\partial \widehat{x}_j(\Lambda)}{\partial \lambda_q}\left[\sigma_j - Q\,(\tau, \widehat{x}_j(\Lambda)) - \sum_{i=1}^{m}\alpha_{ij}\lambda_i\right] + \beta_q +$$

$$+Q\left(\tau,\lambda_q\right)-\sum_{j=1}^{n}\alpha_{qj}\widehat{x}_j(\Lambda)=0\qquad q=\overline{1,m}\,,\tag{2.2.6}$$

since $j=\overline{1,n}$ and $q=\overline{1,m}$:

$$\frac{\partial R}{\partial s}(\tau,\widehat{x}_j(\Lambda))=Q\left(\tau,\widehat{x}_j(\Lambda)\right)\qquad\text{and}\qquad\frac{\partial R}{\partial s}(\tau,\lambda_q)=Q\left(\tau,\lambda_q\right)\,.$$

Finally, by using (2.2.4), equality (2.2.6) can be simplified to

$$\beta_q+Q\big(\tau,\widehat{\widehat{\lambda}}_q\big)-\sum_{j=1}^{n}\alpha_{qj}\widehat{x}_j(\widehat{\widehat{\Lambda}})=0\qquad q=\overline{1,m}\,.\tag{2.2.7}$$

Here the vector $\widehat{\widehat{\Lambda}}=\underset{\lambda}{\operatorname{argmin}}\,U(\tau,\widehat{X}(\Lambda),\Lambda)$ has non-negative components. Then the equality

$$\min_{\lambda}\max_{x}U(\tau,X,\Lambda)=U\Big(\tau,\widehat{X}(\widehat{\widehat{\Lambda}}),\widehat{\widehat{\Lambda}}\Big)$$

is true, where $\left\|\widehat{\widehat{\Lambda}}\right\|=\left\|\begin{array}{cccc}\widehat{\widehat{\lambda}}_1 & \widehat{\widehat{\lambda}}_2 & \dots & \widehat{\widehat{\lambda}}_m\end{array}\right\|^{\mathrm{T}}$. It *exists and is unique* by virtue of the strict convexity in $\Lambda$ of the function $U(\tau,X,\Lambda)$.

Let us now find *the value* of the minimum in $\Lambda$ for the function $U(\tau,\widehat{X}(\Lambda),\Lambda)$. We have

$$U(\tau,\widehat{X}(\widehat{\widehat{\Lambda}}),\widehat{\widehat{\Lambda}})=\sum_{j=1}^{n}\Big(\sigma_j\widehat{x}_j(\widehat{\widehat{\Lambda}})-R(\tau,\widehat{x}_j(\widehat{\widehat{\Lambda}}))\Big)+$$

$$+\sum_{i=1}^{m}\Big(\beta_i\widehat{\widehat{\lambda}}_i+R(\tau,\widehat{\widehat{\lambda}}_i)\Big)-\sum_{j=1}^{n}\sum_{i=1}^{m}\alpha_{ij}\widehat{x}_j(\widehat{\widehat{\Lambda}})\widehat{\widehat{\lambda}}_i\,.$$

Regrouping the terms, we get that

$$U(\tau,\widehat{X}(\widehat{\widehat{\Lambda}}),\widehat{\widehat{\Lambda}})=$$

$$=\sum_{i=1}^{m}\Big(\beta_i\widehat{\widehat{\lambda}}_i+R(\tau,\widehat{\widehat{\lambda}}_i)\Big)+\sum_{j=1}^{n}\Big(\sigma_j-\sum_{i=1}^{m}\alpha_{ij}\widehat{\widehat{\lambda}}_i\Big)\widehat{x}_j(\widehat{\widehat{\Lambda}})-\sum_{j=1}^{n}R(\tau,\widehat{x}_j(\widehat{\widehat{\Lambda}}))\,.$$

Finally, by virtue of (2.2.4), we obtain

$$U(\tau,\widehat{X}(\widehat{\widehat{\Lambda}}),\widehat{\widehat{\Lambda}})=\min_{\lambda}\max_{x}U(\tau,X,\Lambda)=$$

$$= \sum_{i=1}^{m} \left( \beta_i \widehat{\widehat{\lambda}}_i + R(\tau, \widehat{\widehat{\lambda}}_i) \right) + \sum_{j=1}^{n} \left( \widehat{x}_j(\widehat{\widehat{\Lambda}}) Q(\tau, \widehat{x}_j(\widehat{\widehat{\Lambda}})) - R(\tau, \widehat{x}_j(\widehat{\widehat{\Lambda}})) \right). \tag{2.2.8}$$

3°. Now we find $\max\limits_{x} \min\limits_{\lambda} U(\tau, X, \Lambda)$ by reasoning similar to that in 2°.

Since the function $U(\tau, X, \Lambda)$ is strictly convex in the set of components of the vector $\Lambda$, then a sufficient condition for its minimum in $\Lambda$ for a fixed $X$ will be

$$-Q(\tau, \widehat{\lambda}_i(X)) = \beta_i - \sum_{j=1}^{n} \alpha_{ij} x_j \quad i = \overline{1, m}\,. \tag{2.2.9}$$

Minimum value of the function $U(\tau, X, \Lambda)$ by $\Lambda$ for fixed $\tau$ and $X$ is

$$U(\tau, X, \widehat{\Lambda}(X)) = \sum_{j=1}^{n} \left( \sigma_j x_j - R(\tau, x_j)) \right) +$$

$$+ \sum_{i=1}^{m} \left( \beta_i \widehat{\lambda}_i(X) + R(\tau, \widehat{\lambda}_i(X)) \right) - \sum_{j=1}^{n} \sum_{i=1}^{m} \alpha_{ij} x_j \widehat{\lambda}_i(X)\,.$$

Due to the strict upward convexity of this function in $X$, a sufficient condition for its maximum in $X$ (after rearranging the terms) will be:

$$\sum_{i=1}^{m} \frac{\partial \widehat{\lambda}_i(X)}{\partial x_k} \left[ \beta_i + Q\left(\tau, \widehat{\lambda}_i(X)\right) - \sum_{j=1}^{n} \alpha_{ij} x_j \right] + \sigma_k -$$

$$-Q\left(\tau, x_k\right) - \sum_{i=1}^{m} \alpha_{ik} \widehat{\lambda}_i(X) = 0 \qquad k = \overline{1, n}\,. \tag{2.2.10}$$

The expressions in square brackets in (2.2.10) are equal to zero by (2.2.9). Therefore, the sufficient condition for the maximum of $U(\tau, X, \widehat{\Lambda}(X))$ with respect to $X$ takes the form:

$$\sigma_k - Q\left(\tau, \widehat{\widehat{x}}_k\right) - \sum_{i=1}^{m} \alpha_{ik} \widehat{\lambda}_i(\widehat{\widehat{X}}) = 0 \qquad k = \overline{1, n}\,. \tag{2.2.11}$$

Here the vector $\widehat{\widehat{X}} = \underset{x}{\operatorname{argmax}}\, U(\tau, X, \widehat{\Lambda}(X))$ has non-negative components. It *exists and is unique* due to the strict concavity with respect to $X$. Finally, the equality

$$\max_{x} \min_{\lambda} U(\tau, X, \Lambda) = U(\tau, \widehat{\widehat{X}}, \widehat{\Lambda}(X)),$$

is valid for the function $U(\tau, X, \Lambda)$.

For *value* of the maximum of the function $U\left(\tau, X, \widehat{\Lambda}(X)\right)$ with respect to $X$ (after rearranging the terms and taking into account (2.2.11)) we have

$$U(\tau, \widehat{\widehat{X}}, \widehat{\Lambda}(\widehat{\widehat{X}})) = \max_{x} \min_{\lambda} U(\tau, X, \Lambda) =$$

$$= \sum_{i=1}^{m} \left(\beta_i \widehat{\lambda}_i(\widehat{\widehat{X}}) + R(\tau, \widehat{\lambda}_i(\widehat{\widehat{X}}))\right) + \sum_{j=1}^{n} \left(\widehat{\widehat{x}}_j Q(\tau, \widehat{\widehat{x}}_j) - R(\tau, \widehat{\widehat{x}}_j)\right). \tag{2.2.12}$$

4°. Now note that from conditions (2.2.7) and (2.2.9) follows the equality $\widehat{\widehat{\Lambda}} = \widehat{\Lambda}\big(\widehat{\widehat{X}}\big)$, and from conditions (2.2.4) and (2.2.11) follows the equality $\widehat{\widehat{X}} = \widehat{X}\big(\widehat{\widehat{\Lambda}}\big)$.

Indeed, condition (2.2.7) is a sufficient condition for the maximum of the function $U(\tau, X, \widehat{\Lambda}(X))$ with respect to $X$, and (2.2.11) is a sufficient condition for the minimum of $U(\tau, \widehat{X}(\Lambda), \Lambda)$ with respect to the components of the vector $\Lambda$.

Let us check, for example, the equality $\widehat{\widehat{X}} = \widehat{X}\big(\widehat{\widehat{\Lambda}}\big)$. To do this, we rewrite conditions (2.2.4) and (2.2.11) for $j = \overline{1,n}$ replacing the index $k$ in (2.2.11) with $j$.

$$\begin{aligned} -Q\left(\tau, \widehat{x}_j\big(\Lambda\big)\right) &= -\sigma_j + \sum_{i=1}^{m} \alpha_{ij}\lambda_i \qquad j = \overline{1,n}, \\ -Q\big(\tau, \widehat{\widehat{x}}_j\big) &= -\sigma_j + \sum_{i=1}^{m} \alpha_{ij}\widehat{\lambda}_i\big(\widehat{\widehat{X}}\big) \quad j = \overline{1,n}. \end{aligned} \tag{2.2.13}$$

The first group of these equalities is true at any values of numbers $\lambda_i > 0 \quad i = \overline{1,m}$, including $\lambda_i = \widehat{\lambda}_i\big(\widehat{\widehat{X}}\big) \quad i = \overline{1,m}$. Then the right-hand sides of equalities (2.2.13) are the same. This means that the left-hand sides will also be equal, i.e.

$$Q\big(\tau,\widehat{x}_j\big(\widehat{\widehat{\Lambda}}\big)\big)=Q\big(\tau,\widehat{\widehat{x}}_j\big)\qquad j=\overline{1,n}\,.$$

From this it follows that the equalities $\widehat{x}_j(\widehat{\widehat{\Lambda}})=\widehat{\widehat{x}}_j\quad j=\overline{1,n}$, by virtue of the monotonicity in $s$ of the function $Q(\tau,s)$. That is, $\widehat{\widehat{X}}=\widehat{X}(\widehat{\widehat{\Lambda}})\,.$

Substituting $\widehat{\Lambda}\big(\widehat{\widehat{X}}\big)=\widehat{\widehat{\Lambda}}$ and $\widehat{X}\big(\widehat{\widehat{\Lambda}}\big)=\widehat{\widehat{X}}$ in (2.2.8) and (2.2.12), we obtain that

$$\min_{\lambda}\max_{x}U(\tau,X,\Lambda)=\max_{x}\min_{\lambda}U(\tau,X,\Lambda)\,. \tag{2.2.14}$$

Equality (2.2.14) means that the function $U(\tau,X,\Lambda)\quad\forall\tau>0$ *has a unique stationary point* $\{\widehat{\widehat{X}};\,\widehat{\widehat{\Lambda}}\}$, which, due to the continuous differentiability of $U(\tau,X,\Lambda)$, is *stationary* for this function. The stationarity conditions can be written as

$$\begin{cases}\operatorname*{grad}_{x}U=o\,,\\ \operatorname*{grad}_{\lambda}U=o\,,\end{cases}$$

equivalent to system (1.2.7), which proves the conclusion of the theorem.

`The theorem is proved.`

Other properties of the $U$-function, useful in solving practical problems, are described by the following theorems.

`Theorem 2.2.3.` **If $L(X,\Lambda)$ is the Lagrange function of the pair of problems** (1.2.1) **and** (1.2.2)**, then** $\forall X$ **and** $\forall\Lambda$

$$\lim_{\tau\to+0}U(\tau,X,\Lambda)=L(X,\Lambda)\,. \tag{2.2.15}$$

`Proof.`

The Lagrange function of problems (1.2.1) and (1.2.2) has the form

$$L(X,\Lambda)=\sum_{j=1}^{n}\sigma_jx_j-\sum_{i=1}^{m}\lambda_i\Big(-\beta_i+\sum_{j=1}^{n}\alpha_{ij}x_j\Big)\qquad\text{or}$$

$$L(X,\Lambda)=\sum_{j=1}^{n}\sigma_j x_j+\sum_{i=1}^{m}\beta_i\lambda_i-\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}x_j\lambda_i\,.$$

Therefore, the $U$-function can be written as

$$U(\tau,X,\Lambda)=L(X,\Lambda)-\sum_{j=1}^{n}R\big(\tau,x_j\big)+\sum_{i=1}^{m}R\big(\tau,\lambda_i\big)\,. \quad (2.2.16)$$

Applying Lemma 2.1.1 to formula (2.2.16), we obtain the assertion of the theorem being proved.

`The theorem is proved.`

Just like before, for linear problems (1.2.1) and (1.2.2) we will call

the set of points $\{\,\overline{X}(\tau);\,\overline{\Lambda}(\tau)\,\}\quad\forall\tau>0$ of the space $E^n\otimes E^m$ a *stationary trajectory* of the $U$-function for the pair of problems (1.2.1) and (1.2.2).

From Theorem 2.2.1 it follows that there exist vector functions $\overline{X}(\tau)$ and $\overline{\Lambda}(\tau)$, as well as a scalar function $\overline{U}(\tau)=U\big(\tau,\overline{X}(\tau),\overline{\Lambda}(\tau)\big)\quad\forall\tau>0\,.$

The following theorems are also valid for $\overline{X}(\tau)$ and $\overline{\Lambda}(\tau)$.

`Theorem 2.2.4.` **In problems** (1.2.1) **and** (1.2.2)**, having bounded optimal values of the objective functions, the vector functions $\overline{X}(\tau)$ and $\overline{\Lambda}(\tau)$ have bounded components on the set** $\forall\tau>0\,.$

`Proof.`

1°. According to (1.2.6), conditions for the stationarity of the $U$-function can be written in the form

$$\begin{cases}\overline{\lambda}_i(\tau) &= \dfrac{\partial P}{\partial s}\Big(\tau,-\beta_i+\sum\limits_{j=1}^{n}\alpha_{ij}\dfrac{\partial P}{\partial s}\Big(\tau,\quad\sigma_j-\sum\limits_{q=1}^{m}\alpha_{qj}\overline{\lambda}_q(\tau)\Big)\Big)\\ &\quad i=\overline{1,m}\,,\\ \overline{x}_j(\tau) &= \dfrac{\partial P}{\partial s}\Big(\tau,\quad\sigma_j-\sum\limits_{i=1}^{m}\alpha_{ij}\dfrac{\partial P}{\partial s}\Big(\tau,-\beta_i+\sum\limits_{p=1}^{n}\alpha_{ip}\,\overline{x}_p(\tau)\Big)\Big)\\ &\quad j=\overline{1,n}\,.\end{cases} \quad (2.2.17)$$

That is, system (1.2.7) splits into two independent subsystems, the unknowns in the first of which are the components of the vector function $\overline{\Lambda}(\tau)$, and in the second — the components $\overline{X}(\tau)$.

Let us examine in detail, for example, the second of them. We introduce for convenience the following scalar functions

$$V_j\Big(\tau, \overline{X}(\tau)\Big) = \sigma_j - \sum_{i=1}^{m} \alpha_{ij} \frac{\partial P}{\partial s}\Big(\tau, f_i\big(\overline{X}(\tau)\big)\Big) \qquad j = \overline{1,n}\,,$$

$$\text{where} \qquad f_i\big(\overline{X}(\tau)\big) = -\beta_i + \sum_{p=1}^{n} \alpha_{ip}\, \overline{x}_p(\tau) \qquad i = \overline{1,m}\,.$$

In this case, the second group of equations in (2.2.17) takes the form of a system of equalities

$$\overline{x}_j(\tau) = \frac{\partial P}{\partial s}\Big(\tau, V_j\big(\tau, \overline{X}(\tau)\big)\Big) \quad j = \overline{1,n}\,. \tag{2.2.18}$$

The partial derivative of the function $\dfrac{\partial P}{\partial s}\Big(\tau, V_j\big(\tau, X\big)\Big)$ by $x_j$ will be

$$\frac{\partial}{\partial x_j}\frac{\partial P}{\partial s}\Big(\tau, V_j\big(\tau, X\big)\Big) \;=\; \frac{\partial^2 P}{\partial s^2}\Big(\tau, V_j\big(\tau, X\big)\Big) \cdot \frac{\partial V_j}{\partial x_j} \;=$$

$$= \;\frac{\partial^2 P}{\partial s^2}\Big(\tau, V_j\big(\tau, X\big)\Big) \cdot \frac{\partial}{\partial x_j}\Big(\sigma_j - \sum_{i=1}^{m} \alpha_{ij} \frac{\partial P}{\partial s}\Big(\tau, V_i\,(\tau, X)\Big)\Big) =$$

$$= -\frac{\partial^2 P}{\partial\, s^2}\Big(\tau, V_j\big(\tau, X\big)\Big) \cdot \Big(\sum_{i=1}^{m} \alpha_{ij} \frac{\partial^2 P}{\partial\, s^2}(\tau, X)\, \frac{\partial f_i}{\partial x_j}\Big) \;=$$

$$= -\frac{\partial^2 P}{\partial\, s^2}\Big(\tau, V_j\big(\tau, X\big)\Big) \cdot \sum_{i=1}^{m} \alpha_{ij}^2 \frac{\partial^2 P}{\partial\, s^2}(\tau, X)\;.$$

By (1.1.3), the second derivative $\dfrac{\partial^2 P}{\partial\, s^2}(\tau, s)$ is strictly positive $\forall s$. Therefore

$$\frac{\partial}{\partial x_j}\frac{\partial P}{\partial s}\Big(\tau, V_j\big(\tau, X\big)\Big) \;=$$

$$= \quad -\frac{\partial^2 P}{\partial\, s^2}\Big(\tau, V_j\big(\tau, X\big)\Big) \cdot \sum_{i=1}^{m} \alpha_{ij}^2 \frac{\partial^2 P}{\partial\, s^2}(\tau, X) \le 0\,.$$

This inequality will aslo be strict in the case of $\sum\limits_{i=1}^{m} |\,\alpha_{ij}| > 0$ .

Thus, the right-hand side of the $j-$th equation in system (2.2.18) is a monotonically decreasing function of the $j-$th component of the vector $\overline{X}(\tau)$.

2°. Let us now consider the $j$-th equation in system (2.2.18), in which all unknowns are fixed except $\overline{x}_j$. Let it have the form $\overline{x}_j = Z(\overline{x}_j)$. Note that both the domain of the function $Z(\overline{x}_j)$, and its range of values are the set of positive numbers. In this case, it follows from 1° that on the entire domain of definition the continuous function $Z(\overline{x}_j)$ is monotonically decreasing. Then there exists a monotonically decreasing function $T(\overline{x}_j)$, which is inverse for $Z(\overline{x}_j)$.

Let $\overline{x}_j^*$ be the solution of the equation under consideration, i.e. $\overline{x}_j^* = Z(\overline{x}_j^*)$. Obviously, and $T(\overline{x}_j^*) = \overline{x}_j^*$. If $\overline{x}_j^*$ is bounded above, then there exists a number $D_j : +\infty > D_j > 0 : \quad \overline{x}_j^* \le D_j$, and, therefore,

$$\overline{x}_j^* = T(\overline{x}_j^*) = Z(\overline{x}_j^*) \le D_j\,.$$

Then, due to the monotonic decrease $T(\overline{x}_j)$, we get

$$T\Big(Z\left(\overline{x}_j^*\right)\Big) \ge T(D_j)\,.$$

In other words, $\exists\, d_j > 0 : \; d_j \le \overline{x}_j^*$, Where $d_j = T(D_j) \le D_j$. That is, $\overline{x}_j^*$ is also bounded from below by a strictly positive value. Similarly, it can be shown that the boundedness from below of the component $\overline{x}_j^*$ implies its boundedness from above.

3°. Let us now show that the solutions of system (2.2.17) will be bounded $\forall \tau > 0$ in the case of bounded solutions of problems (1.2.1)—(1.2.2). And in the case of unbounded solutions of problems (1.2.1)—(1.2.2), the solutions of system (2.2.17) will be unbounded for $\tau \in (0, +\infty)$.

Indeed, if solutions of problems (1.2.1)—(1.2.2) exist and are finite, then the systems

$$\begin{cases} \sigma_j - \sum\limits_{i=1}^{m} \alpha_{ij}\lambda_i \;\le\; 0 \qquad i = \overline{1,m}\,, \\ \qquad\qquad \lambda_i \;\ge\; 0 \qquad i = \overline{1,m} \end{cases}$$

And

$$\left\{\begin{array}{rcll} -\beta_i + \sum\limits_{j=1}^{n} \alpha_{ij} x_j & \leq & 0 & i = \overline{1,m}\,, \\ x_j & \geq & 0 & j = \overline{1,n} \end{array}\right. \tag{2.2.19}$$

consistent.

Let us consider the second of them. In this case there exists a vector $X_{(j)}$ such that $V_j\big(\tau, X_{(j)}\big) = 0 \;\; \forall j = [1,n]\,.$
Note that each left-hand side in equations (2.2.18), which by Theorem 2.2.1 has a finite value for any fixed $\tau > 0\,$, may be equal to, greater than, or less than, than $\Phi(0) = \dfrac{\partial P}{\partial s}\Big(\tau, 0\Big)\,$. Then, by virtue of item $2^\circ$, it turns out to be bounded both below and above by one of three strictly positive numbers $T\big(\Phi(0)\big),\ \Phi(0)$ or $Z\big(\Phi(0)\big)$, each of which does not depend on $\tau\,$.

For the first of the systems (2.2.19), the arguments are similar.

Now let at least one of the systems (3.19) be inconsistent. Let us assume that this is the second of them. Then at least for one value of the index $i$ there will be $f_i(X) > 0$ for all $X$ with non-negative components. Including for the solution of system (2.2.18) $\overline{X}(\tau)\,$, which exists by Theorem 2.2.1.
But then by virtue of

$$\overline{\lambda}_i(\tau) = \frac{\partial P}{\partial s}\Big(\tau, f_i\big(\overline{X}(\tau)\big)\Big) = \Phi\left(\frac{f_i\big(\overline{X}(\tau)\big)}{\tau}\right)$$

equality will be satisfied $\varliminf\limits_{\tau \to +0} \Phi\Big(\dfrac{1}{\tau} f_i\big(\overline{X}(\tau)\big)\Big) = +\infty\,,$ from which it follows that there are unbounded components of solutions to system (2.2.17).

In the case of incompatibility of the first of the systems (2.2.19), the reasoning is similar.

`The theorem is proved.`

`Theorem 2.2.5` **For the pair of problems** (1.2.1) **and** (1.2.2) **the vector functions** $\overline{X}(\tau)$ **and** $\overline{\Lambda}(\tau)$ **are continuously differentiable** $\forall \tau > 0\,.$

Proof.

The proof repeats the proof of Theorem 2.1.3, if the condition $\forall\tau \in (0;\, \tau_0)$ is replaced by $\forall\tau > 0$. That is, the assumption of the existence of a compact neighborhood of a stationary point is absent here.

The theorem is proved.

Corollary 2.2.1. **On a stationary trajectory for the pair of problems** (1.2.1) **and** (1.2.2) **with bounded solutions there exist finite limits**

$$\lim_{\tau\to+0} \overline{X}(\tau)\,, \qquad \lim_{\tau\to+0} \overline{\Lambda}(\tau)$$

**and**

$$\lim_{\tau\to+0} U\Big(\tau, \overline{X}(\tau), \overline{\Lambda}(\tau)\Big)\,.$$

Proof.

From the continuous differentiability and boundedness of vector-functions $\overline{X}(\tau)$ and $\overline{\Lambda}(\tau)$ $\forall\tau > 0$ in the case of a pair of linear mutually dual problems with bounded solutions follows their uniform continuity. Therefore, the limits specified in the statement of the corollary exist and are finite.

The corollary is proved.

The properties of these limits are described by the following theorems, which are consistent with the theory of linear programming.

Theorem 2.2.6. **On a stationary trajectory for a pair of linear mutually dual problems with bounded solutions of the form** (1.2.1) **and** (1.2.2)**, the following equalities hold**

$$\lim_{\tau\to+0} \overline{\lambda}_i(\tau)\, f_i\big(\overline{X}(\tau)\big) = 0 \qquad i = \overline{1,m}\,,$$

$$\lim_{\tau\to+0} \overline{x}_j(\tau)\, g_j\big(\overline{\Lambda}(\tau)\big) = 0 \qquad j = \overline{1,n}\,,$$

$$\lim_{\tau\to+0} \Big(F(\overline{X}(\tau)) - G(\overline{\Lambda}(\tau))\Big) = 0\,.$$

Proof.

1°. Note first that for finite $s > 0$ the equality

$$\lim_{\tau\to+0} Q(\tau, s) = \lim_{\tau\to+0} \tau\Psi(s) = 0\,.$$

Multiplying both sides of equalities (1.2.7) by $\overline{\lambda}_i \quad i = \overline{1,m}$ and, accordingly, $\overline{x}_j \quad j = \overline{1,n}$ we arrive at the relations

$$\overline{\lambda}_i(\tau)\, f_i\big(\overline{X}(\tau)\big) = \lambda_i(\tau)\, Q\big(\tau, \lambda_i(\tau)\big) \qquad i = \overline{1,m}\,,$$

$$\overline{x}_j(\tau)\, g_j\big(\overline{\Lambda}(\tau)\big) = x_j(\tau)\, Q\big(\tau, x_j(\tau)\big) \qquad j = \overline{1,n}\,.$$

Then, by virtue of the above remark, we arrive at the first two statements of the lemma.

2°. The stationarity conditions for the $U$-function imply the equalities

$$\sigma_j = Q\left(\tau, \overline{x}_j(\tau)\right) + \sum_{i=1}^{m} \alpha_{ij} \overline{\lambda}_i(\tau) \qquad j = \overline{1,n}\,,$$

due to which

$$\sum_{j=1}^{n} \sigma_j \overline{x}_j(\tau) = \sum_{j=1}^{n} \overline{x}_j(\tau)\, Q\left(\tau, \overline{x}_j(\tau)\right) + \sum_{j=1}^{n} \sum_{i=1}^{m} \alpha_{ij} \overline{\lambda}_i(\tau) \overline{x}_j(\tau)\,.$$

Similarly we find that

$$\sum_{i=1}^{m} \beta_i \overline{\lambda}_i(\tau) = -\sum_{i=1}^{m} \overline{\lambda}_i(\tau)\, Q\left(\tau, \overline{\lambda}_i(\tau)\right) + \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} \overline{x}_j(\tau) \overline{\lambda}_i(\tau)\,.$$

Termwise subtraction of the last two equalities $\forall \tau > 0$ gives

$$F\big(\overline{x}_j(\tau)\big) - G\big(\overline{\lambda}_i(\tau)\big) =$$

$$= \sum_{j=1}^{n} \overline{x}_j(\tau)\, Q\big(\tau, \overline{x}_j(\tau)\big) + \sum_{i=1}^{m} \overline{\lambda}_i(\tau)\, Q\big(\tau, \overline{\lambda}_i(\tau)\big).$$

Then on the stationary trajectory, by virtue of 1°, the equality will also be true

$$\lim_{\tau \to +0} \Big( F(\overline{X}(\tau)) - G(\overline{\Lambda}(\tau)) \Big) = 0\,.$$

The theorem is proved.

Theorem 2.2.7. **On stationary trajectories for the pair of problems** (1.2.1) **and** (1.2.2)

$$\lim_{\tau \to +0} U\Big(\tau, \overline{X}(\tau), \overline{\Lambda}(\tau)\Big) = F\left(X^*\right) = G\left(\Lambda^*\right), \quad (2.2.21)$$

**and in the case of unique solutions to the pair of problems** (1.2.1) **and** (1.2.2) **the equalities**

$$\lim_{\tau\to+0}\overline{X}(\tau)=X^*\quad\textbf{and}\quad\lim_{\tau\to+0}\overline{\Lambda}(\tau)=\Lambda^*\,. \quad (2.2.22)$$

Proof.

On a stationary trajectory $\forall\tau>0$ in force (2.2.16)

$$U\Big(\tau,\overline{X}(\tau),\overline{\Lambda}(\tau)\Big)=L\Big(\overline{X}(\tau),\overline{\Lambda}(\tau)\Big)-$$

$$-\sum_{j=1}^{n}R\Big(\tau,\overline{x}_j(\tau)\Big)+\sum_{i=1}^{m}R\Big(\tau,\overline{\lambda}_i(\tau)\Big)\,,$$

so $\forall\tau>0$ fair estimates:

$$U\Big(\tau,\overline{X}(\tau),\overline{\Lambda}(\tau)\Big)\le L\Big(\overline{X}(\tau),\overline{\Lambda}(\tau)\Big)+\sum_{i=1}^{m}R\Big(\tau,\overline{\lambda}_i(\tau)\Big)\le$$

$$\le G^*+m\max_{i=[1,m]}R\Big(\tau,\overline{\lambda}_i(\tau)\Big); \quad (2.2.23)$$

$$U\Big(\tau,\overline{X}(\tau),\overline{\Lambda}(\tau)\Big)\ge L\Big(\overline{X}(\tau),\overline{\Lambda}(\tau)\Big)-\sum_{j=1}^{n}R\Big(\tau,\overline{x}_j(\tau)\Big)\ge$$

$$\ge F^*-n\max_{j=[1,n]}R\Big(\tau,\overline{x}_j(\tau)\Big)\,. \quad (2.2.24)$$

By Theorem 2.2.4, the values of the vector functions $\overline{X}(\tau)$ and $\overline{\Lambda}(\tau)$ are bounded on the stationary trajectory. Then from Corollary 2.2.1 and Lemma 1.1.1 we obtain that

$$\lim_{\tau\to+0}\max_{i=[1,m]}R\Big(\tau,\overline{\lambda}_i(\tau)\Big)\;=\;\lim_{\tau\to+0}\max_{j=[1,n]}R\Big(\tau,\overline{x}_j(\tau)\Big)\;=\;0$$

and there is a limit (2.2.21)

$$\lim_{\tau\to+0}U\Big(\tau,\overline{X}(\tau),\overline{\Lambda}(\tau)\Big)=F^*=G^*.$$

Under the above assumptions, the vector functions $\overline{X}(\tau)$ and $\overline{\Lambda}(\tau)$ are continuous $\forall\tau>0$. From their continuity and the properties of superposition of continuous functions, we obtain that there exist limits

$$\lim_{\tau\to+0}L\Big(\overline{X}(\tau),\overline{\Lambda}(\tau)\Big)=F^*=G^*\,,$$

$$F\Big(\lim_{\tau\to+0}\overline{X}(\tau)\Big)=F^*\qquad\text{and}\qquad G\Big(\lim_{\tau\to+0}\overline{\Lambda}(\tau)\Big)=G^*\,.$$

Finally, in the regular case of problems (1.2.1) and (1.2.2), the Lagrange function has a unique stationary point $\{X^*,\Lambda^*\}$, therefore equalities (2.2.22) are also valid.

`The theorem is proved.`

Note also that the stationary point of a $U$-function always exists and is locally unique. This is true, for example, for linear programming problems with more than one optimal point.

In conclusion, we will give examples of solutions by the feedback function method for mutually dual pairs of linear programming problems with different types of solutions.

It is known from theory [1, 9] that only the following three cases are possible:

1°. Both problems have a bounded solution.

2°. One of the problems is incompatible, and the other has an unbounded solution.

3°. Both problems are incompatible.

Case 1° was illustrated in § 1.2 with problems 1.2.1 and 1.2.2. Here we use the feedback function of the form $Q(\tau,s)=\tau\left(s-\dfrac{1}{s}\right)$ again.

Case 2° is illustrated with

`Problem 2.2.1.` For the pair of problems:

**primary problem**:
*maximize in* $E^2$ *the function* $F(x)=2x_1$,
*subject to*
$x_1\ge 0\,,\ x_2\ge 0$ *and*
$-x_1+x_2\le 1\,;$

**dual problem**:
*minimize in* $E^2$ *the function* $G(\Lambda)=\lambda\,,$
*subject to* $\lambda\ge 0$ *and*
$-\lambda\ge 2\,;\quad \lambda\ge 0\,.$

For the primal problem, the value of $x_1$ can be arbitrarily large in absolute value. The dual problem is inconsistent here.

System (1.2.7) for this case has the form

$$\left\{\begin{array}{rcl} 2+\overline{\lambda} & = & -\tau\left(\overline{x}_1-\dfrac{1}{\overline{x}_1}\right), \\ -\overline{\lambda} & = & -\tau\left(\overline{x}_2-\dfrac{1}{\overline{x}_2}\right), \\ 1+\overline{x}_1-\overline{x}_2 & = & \tau\left(\overline{\lambda}-\dfrac{1}{\overline{\lambda}}\right). \end{array}\right. \tag{2.2.25}$$

Its solutions for various values of the parameter $\tau$ are given in Table 2.2.1a and Table 2.2.1b. For brevity, the following notation is used:

$$\begin{array}{rcl} F_\tau & = & 2\overline{x}_1\,, \\ f_\tau & = & 1-\overline{x}_1+\overline{x}_2\,, \\ G_\tau & = & \overline{\lambda}\,, \\ g_{1\tau} & = & 2+\overline{\lambda}\,, \\ g_{2\tau} & = & -\overline{\lambda}\,. \end{array}$$

T a b l e 2.2.1a.
Solutions of system (2.2.25) for problem 2.2.1

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $F_\tau$ | $f_\tau$ |
|---|---|---|---|---|
| $10^{-1}$ | 9.604136568 | 10.594374945 | 19.208273137 | -0.009761623 |
| $10^{-2}$ | 99.510049126 | 100.509949374 | 199.020098251 | -0.000099751 |
| $10^{-3}$ | 999.501000499 | 1000.500999000 | 1999.002001000 | -0.000001000 |
| $10^{-4}$ | 9999.500100000 | 10000.500100000 | 19999.000200000 | -0.000000010 |
| $10^{-5}$ | 99999.500010000 | 100000.500000000 | 199999.000000000 | 0.000000000 |
| $10^{-6}$ | 999999.500000000 | 1000000.500000000 | 1999999.000000000 | 0.000000000 |
| $10^{-7}$ | 9999999.500000000 | 10000000.500000000 | 19999999.000000000 | 0.000000000 |
| $10^{-8}$ | 99999999.500000000 | 100000000.500000000 | 199999999.000000000 | 0.000000000 |

The obtained results show that the constraint $f_\tau(x)\le 0$ is active in the primary problem. The constraints $g_{1\tau}(\lambda)\ge 0$ and $g_{2\tau}(\lambda)\ge 0$ are active in the dual problem. The variables $\overline{x}_1$ and $\overline{x}_2$ increase without bound if $\tau\to+0$.

T a b l e 2.2.1b.
Solutions of system (2.2.25) for problem 2.2.1

| $\tau$ | $\overline{\lambda}$ | $G_\tau$ | $g_{1\tau}$ | $g_{2\tau}$ |
|---|---|---|---|---|
| $10^{-1}$ | -1.049998523 | -1.049998523 | 0.950001477 | 1.049998523 |
| $10^{-2}$ | -1.005000001 | -1.005000001 | 0.994999999 | 1.005000001 |
| $10^{-3}$ | -1.000500000 | -1.000500000 | 0.999500000 | 1.000500000 |
| $10^{-4}$ | -1.000050000 | -1.000050000 | 0.999950000 | 1.000050000 |
| $10^{-5}$ | -1.000005000 | -1.000005000 | 0.999995000 | 1.000005000 |
| $10^{-6}$ | -1.000000500 | -1.000000500 | 0.999999500 | 1.000000500 |
| $10^{-7}$ | -1.000000050 | -1.000000050 | 0.999999950 | 1.000000050 |
| $10^{-8}$ | -1.000000005 | -1.000000005 | 0.999999995 | 1.000000005 |

Finally, case 3° is illustrated with

`Problem 2.2.2`. For the pair of problems: **primary problem**:

*maximize in* $E^2$ *the function* $F(x) = 2x_1 + 3x_2$,
*subject to*
$x_1 \geq 0\ ,\ x_2 \geq 0$ *and*
$x_1 - x_2 \leq 1\ ,\ -x_1 + x_2 \leq -5\ ;$

**dual problem**:
*minimize in* $E^2$ *the function* $G(\lambda) = \lambda_1 - 5\lambda_2$,
*subject to* $\lambda_1 \geq 0\ ,\ \lambda_2 \geq 0$ *and*
$\lambda_1 - \lambda_2 \geq 2\ ,\ -\lambda_1 + \lambda_2 \geq 3\,.$

Here both problems are incompatible. Its solutions are given in Table 2.2.2a and Table 2.2.2b for various values of the parameter $\tau$. For brevity, we use the following notation in these tables:

$$\begin{aligned} F_\tau &= 2\overline{x}_1 + 3\overline{x}_2\,, \\ f_{1\tau} &= -1 + \overline{x}_1 - \overline{x}_2\,, \\ f_{2\tau} &= 5 - \overline{x}_1 + \overline{x}_2\,, \\ G_\tau &= \overline{\lambda}_1 - 5\overline{\lambda}_2\,, \\ g_{1\tau} &= -2 + \overline{\lambda}_1 - \overline{\lambda}_2\,, \\ g_{2\tau} &= -3 - \overline{\lambda}_1 + \overline{\lambda}_2\,. \end{aligned}$$

T a b l e 2.2.2a.
Solutions of system (2.2.26) for problem 2.2.2

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $F_\tau$ | $f_{1\tau}$ | $f_{2\tau}$ |
|---|---|---|---|---|---|
| $10^{-1}$ | 26.523821049 | 23.556332339 | 123.716639117 | 1.967488710 | 2.032511290 |
| $10^{-2}$ | 251.502712579 | 248.505287580 | 1248.521288000 | 1.997424999 | 2.002575001 |
| $10^{-3}$ | 2501.500275000 | 2498.500525000 | 12498.502130000 | 1.999749250 | 2.000250750 |
| $10^{-4}$ | 25001.500030000 | 24998.500050000 | 124998.500200000 | 1.999974993 | 2.000025007 |
| $10^{-5}$ | 250001.500000000 | 249998.500000000 | 1249998.500000000 | 1.999997500 | 2.000002500 |
| $10^{-6}$ | 2500001.500000000 | 2499998.500000000 | 12499998.500000000 | 1.999999750 | 2.000000250 |
| $10^{-7}$ | 25000001.500000000 | 24999998.500000000 | 124999998.500000000 | 1.999999974 | 2.000000026 |
| $10^{-8}$ | 250000001.500000000 | 250000001.500000000 | 1249999999.000000000 | 2.000000000 | 2.000000000 |

It follows from the obtained results that the constraints $f_{1\tau} \le 0$ and $f_{2\tau} \le 0$ are active in the primary problem. The constraints $g_{1\tau}(\lambda) \ge 0$ and $g_{2\tau}(\lambda) \ge 0$ are active in the dual problem.

The variables $\overline{x}_1$ and $\overline{x}_2$, as well as the variables $\overline{\lambda}_1$ and $\overline{\lambda}_2$, increase without bound if $\tau \to +0$.

System (1.2.7) for this case takes the form

$$\begin{cases} 1 + \overline{x}_1 - \overline{x}_2 &= \tau\left(\overline{\lambda}_1 - \dfrac{1}{\overline{\lambda}_1}\right), \\ 5 - \overline{x}_1 + \overline{x}_2 &= \tau\left(\overline{\lambda}_2 - \dfrac{1}{\overline{\lambda}_2}\right), \\ -2 + \overline{\lambda}_1 - \overline{\lambda}_2 &= -\tau\left(\overline{x}_1 - \dfrac{1}{\overline{x}_1}\right), \\ -3 - \overline{\lambda}_1 + \overline{\lambda}_2 &= -\tau\left(\overline{x}_2 - \dfrac{1}{\overline{x}_2}\right). \end{cases} \tag{2.2.26}$$

T a b l e 2.2.2b.
Solutions of system (2.2.26) for problem 2.2.2

| $\tau$ | $\overline{\lambda}_1$ | $\overline{\lambda}_2$ | $G_\tau$ | $g_{1\tau}$ | $g_{2\tau}$ |
|---|---|---|---|---|---|
| $10^{-1}$ | 19.725582687 | 20.374194596 | -82.145390291 | -2.648611909 | -2.351388091 |
| $10^{-2}$ | 199.747506201 | 200.262493566 | -801.564961628 | -2.514987365 | -2.485012635 |
| $10^{-3}$ | 1999.749750000 | 2000.251250000 | -8001.506500000 | -2.501499875 | -2.498500125 |
| $10^{-4}$ | 19999.749980000 | 20000.250120000 | -80001.500650000 | -2.500149999 | -2.499850001 |
| $10^{-5}$ | 199999.750000000 | 200000.250000000 | -800001.500100000 | -2.500015000 | -2.499985000 |
| $10^{-6}$ | 1999999.750000000 | 2000000.250000000 | -8000001.500000000 | -2.500001500 | -2.499998500 |
| $10^{-7}$ | 19999999.750000000 | 20000000.250000000 | -80000001.500000000 | -2.500000149 | -2.499999851 |
| $10^{-8}$ | 199999999.700000000 | 200000000.300000000 | -800000001.500000000 | -2.500000030 | -2.499999970 |

## 2.3. Smoothing property of feedback functions

Theorem 2.1.3 states that vector functions $\overline{X}(\tau)$ and $\overline{\Lambda}(\tau)$ approximate solutions of problem (1.1.1). They are continuously differentiable with respect to the *instrumental* parameter $\tau$.

We now consider problem (0.1.5) containing in its formulation numerical *exogenous* parameters.

It can be shown that in this case stationary points of the $U$-function are also continuously differentiable functions of parameters of *exogenous* type. Which, in turn, allows using the method of feedback functions to solve optimization problems more general than problem (1.1.1).

Indeed, let $V = \| v_1\, v_2\, \ldots\, v_K \|^{\mathrm{T}} \in E^K$ be a vector of numerical exogenous parameters of the problem:

$$\begin{array}{l} \textit{maximize with respect to } X \in E^n \quad \textit{function } F(X,V) \\ \textit{on the set } V \in \Upsilon,\ \textit{where } \Upsilon \textit{ is compact in } E^K, \\ \textit{under conditions}\colon \quad X \in \Theta_v : \left\{ X \;\middle|\; f_i(X,V) \le 0 \quad i = \overline{1,m} \right\}. \end{array} \tag{2.3.1}$$

**Definition 2.3.1** *A solution* to problem (2.3.1) will be any explicit description *of the dependence* of the vectors $\operatorname{argmax}_{x\in\Theta_v} F(X,V)$ on the exogenous parameters $V \in \Upsilon$.

For brevity, we will denote this dependence as $X_v^*$.

This definition illustrates an easily verifiable

**Example 2.3.1:**

$$\begin{array}{l} \textit{maximize with respect to } x \quad \text{function } v_1 x \\ \textit{under the conditions: } x \geq 0\,, \quad x \leq 5\,, \quad x \leq 5v_2 \\ \textit{for} \quad \Upsilon = \quad \Big\{ v_1 \geq 0\,;\ v_2 \in \mathbb{R} \Big\}. \end{array}$$

A description of its solution can be as follows:

$$\text{in case } v_1 > 0 \quad x_v^* = \begin{cases} \emptyset & \text{at} \quad -\infty < v_2 < 0, \\ 5v_2 & \text{at} \quad 0 \leq v_2 \leq 1, \\ 5 & \text{at} \quad v_2 > 1, \end{cases}$$

$$\text{for } v_1 = 0 \quad x_v^* = \begin{cases} \emptyset & \text{if} \quad -\infty < v_2 < 0, \\ [0, 5v_2] & \text{if} \quad 0 \leq v_2 \leq 1, \\ [0,\ \ 5] & \text{if} \quad v_2 > 1. \end{cases}$$

A graphical representation of this solution is shown in Fig. 2.3.1A.

Here it is necessary to make an important remark: in addition to the parameters, the presence of constraints of the «inequality» type is also essential.

This fact is the cause of the following features of $X_v^*$:

— the domain of definition of the dependence $X_v^*$ may *be narrower* than the domain of definition with respect to $V$ for the functions $F(X,V)$ and $f_i(X,V)$ $i = \overline{1,m}$,

— the dependence $X_v^*$ may be *non-functional* (*ambiguous*),

— finally, even if $X_v^*$ is a function of $V$, it may not have derivatives with respect to the components of $V$.

The above properties complicate the procedures for solving both problems of type 2.3.1 and problems using $X_v^*$.

For illustration, we note that in example 2.3.1 (see Fig. 2.3.1A):

for $v_2 < 0$ the dependence $x_v^*$ is not defined, although all functions in the example condition for such values of the parameter $v_2$ exist.

For $v_1 > 0$ the derivative at the point $v_2 = 1$ for $x_v^*$ with respect to $v_2$ does not exist.

For $v_1 = 0$ at $v_2 > 0$ the dependence $x_v^*$ is defined on ambiguously, that is, it is not a function.

By now, a significant number of algorithms have been developed for solving optimization problems with exogenous parameters. For example, There are methods of *non-differentiable optimization* [2, 6, 13] or *sensitivity theory* [16]. These algorithms make it possible to overcome the computational difficulties generated by the specific properties of the dependences $X_v^*$.

However, classical methods for solving problems of the form (2.3.1) are also of practical interest. These methods based on expansions according to *Taylor's formula*.

Later in this section, we will consider one of such algorithms. It uses feedback functions to construct a *smooth function* $\overline{X}(\tau, V)$ to approximate the dependence $X_v^*$. That is, the limit equality

$$\lim_{\tau \to +0} F\big(\overline{x}(\tau, V), V\big) = F(X_v^*, V)$$

must be satisfied at $\forall V \in \Upsilon$. In the case when $X_v^*$ is a vector-valued function, this equality is simplified to $\lim\limits_{\tau \to +0} \overline{X}(\tau, V) = X_v^*$.

Specifically, $\overline{X}(\tau, V)$ is proposed as such a function. It is a stationary point of the auxiliary function (2.1.2). Indeed, in this case the vector function $\overline{X}(\tau, V)$ exists and is smooth $\forall V \in \Upsilon$.

This allows us to talk about the *smoothing property* of the feedback function method.

Note that the approximating vector function $\overline{X}(\tau, V)$ is defined *implicitly*. This creates additional computational problems. However, the *implicit function system theorem* [6] allows us to overcome them by using Taylor polynomials instead of $\overline{X}(\tau, V)$ .

Let us consider this scheme for solving problem (2.3.1) with a finite value of $F_v^*$. We will assume that all formulated conditions are satisfied $\forall V \in \Upsilon$.

Let us construct the auxiliary function

$$U(\tau, X, \Lambda, V) \;=\; L(X, \Lambda, V) + W(\tau, \Lambda)\,, \;\; \text{where} \;\; W(\tau, \Lambda) \;=\; \sum_{i=1}^{m} R(\tau, \lambda_i), \tag{2.3.2}$$

for problem (2.3.1) and consider its properties.

Under the assumptions made and a fixed vector of parameters $V \in \Upsilon$ the following statements are true (their proofs are similar to those given in § 2.1).

**Theorem 2.3.1.** *$\forall \tau \in (0, \tau_0]$ the function $U(\tau, X, \Lambda, V)$ has a unique stationary point $\{\overline{X}(\tau, V); \overline{\Lambda}(\tau, V)\}$ inside the compact $\Omega_x \otimes \Omega_\Lambda$ , where the vectors $\overline{X}(\tau, V)$ and $\overline{\Lambda}(\tau, V)$ are solutions of the system of equations*

$$\begin{cases} \operatorname*{grad}_x U(\tau, \overline{X}, \overline{\Lambda}, V) = o, \\ \operatorname*{grad}_\lambda U(\tau, \overline{X}, \overline{\Lambda}, V) = o. \end{cases} \tag{2.3.3}$$

The system of equations (2.3.3) has the form

$$\begin{cases} \dfrac{\partial F}{\partial x_j} - \displaystyle\sum_{i=1}^{m} \overline{\lambda}_i(\tau, V) \dfrac{\partial f_i}{\partial x_j} \;=\; 0\,, & j = \overline{1, n}\,, \\ f_i\Big(\overline{X}(\tau, V)\Big) \;=\; Q\Big(\tau, \overline{\lambda}_i(\tau, V)\Big)\,, & i = \overline{1, m}\,, \end{cases} \tag{2.3.4}$$

for problem (2.3.1).

The system (2.3.4) is similar in structure to the conditions of the Karush-Kuhn-Tucker theorem. At the same time, it does not explicitly contain the conditions of non-negativity of the Lagrange multipliers and the conditions of complementary slackness.

Vector-functions $\overline{X}(\tau, V)$ and $\overline{\Lambda}(\tau, V)$ define in $\Omega_x \otimes \Omega_\Lambda$ parametrically (with parameter $\tau$) a line. We call the line as *the stationary trajectory of the problem* (2.3.1).

The following theorems are valid on a stationary trajectory.

**Theorem 2.3.2.** *On the stationary trajectory of the problem* (2.3.1)

$$\lim_{\tau \to +0} U\Big(\tau, \overline{X}(\tau, V), \overline{\Lambda}(\tau, V), V\Big) = F_v^*\,. \tag{2.3.5}$$

*In the case of local uniqueness of the solution to this problem, the following equalities hold*

$$\lim_{\tau \to +0} \overline{X}(\tau, V) = X_v^* \;\; \textit{and} \;\; \lim_{\tau \to +0} \overline{\Lambda}(\tau, V) = \Lambda_v^*\,. \tag{2.3.6}$$

**Theorem 2.3.3.** *On a stationary trajectory, the vector functions* $\overline{X}(\tau, V)$ *and* $\overline{\Lambda}(\tau, V)$ *are continuously differentiable with respect to all their arguments* $\forall \tau > 0 \ \forall V \in \Upsilon$ .

We find the values of the vector functions $\overline{X}(\tau, V)$ and $\overline{\Lambda}(\tau, V)$ for fixed $V \in \Upsilon$ and positive $\tau$ from (2.3.4). Then we apply to the system (2.3.3) the *theorem on implicit functions* [6]. As a result, $\forall r = \overline{1, K}$ we obtain systems of linear equations

$$\begin{cases} \sum\limits_{j=1}^{n} \dfrac{\partial^2 U}{\partial x_p \partial x_j} \dfrac{\partial \overline{x}_j}{\partial v_r} + \sum\limits_{i=1}^{m} \dfrac{\partial^2 U}{\partial x_p \partial \lambda_i} \dfrac{\partial \overline{\lambda}_i}{\partial v_r} = -\dfrac{\partial^2 U}{\partial x_p \partial v_r} & p = \overline{1, n}, \\ \sum\limits_{j=1}^{n} \dfrac{\partial^2 U}{\partial \lambda_q \partial x_j} \dfrac{\partial \overline{x}_j}{\partial v_r} + \sum\limits_{i=1}^{m} \dfrac{\partial^2 U}{\partial \lambda_q \partial \lambda_i} \dfrac{\partial \overline{\lambda}_i}{\partial v_r} = -\dfrac{\partial^2 U}{\partial \lambda_q \partial v_r} & q = \overline{1, m}, \end{cases} \tag{2.3.7}$$

which determine the values of *partial derivatives* of the components of the vector functions $\overline{X}(\tau, V)$ and $\overline{\Lambda}(\tau, V)$ with respect to components of the vector $V$.

If necessary, one can obviously strengthen the assumptions on the properties of the feedback functions, as well as the functions included in the condition of problem (2.3.1). Then, using a similar scheme, one can find higher-order partial derivatives of $\overline{X}(\tau, V)$ and $\overline{\Lambda}(\tau, V)$.

The set of theorems 2.3.1, 2.3.2 and 2.3.3 allows us to use the vector functions $\overline{X}(\tau, V)$ and $\overline{\Lambda}(\tau, V)$ as *approximations* of the dependencies $X_v^*$ and $\Lambda_v^*$. These approximations are smooth and allow for their study the use of computational procedures based on the Taylor formula.

We will illustrate the application of the feedback function method by solving example 2.3.1 with parameters $v_1 \geq 0$ and $v_2 \in \mathbb{R}$.

We will construct a smooth approximation of the $x_v^*$ dependence for example 2.3.1, taking $Q(\tau, s) = \tau \ln s$ as the feedback function. The auxiliary function (2.3.2) here will be

$$U(\tau, x, \lambda_1, \lambda_2, \lambda_3, v_1, v_2) =$$

$$= v_1 x - \lambda_1(-x) - \lambda_2(-5+x) - \lambda_3(-5v_2+x) + R(\tau, \lambda_1) + R(\tau, \lambda_2) + R(\tau, \lambda_3).$$

The condition of its stationarity (system (2.3.4)) has the form

$$\begin{cases} v_1 + \overline{\lambda}_1 - \overline{\lambda}_2 - \overline{\lambda}_3 &= 0, \\ \overline{x} + \tau \ln \overline{\lambda}_1 &= 0, \\ -\overline{x} + 5 + \tau \ln \overline{\lambda}_2 &= 0, \\ -\overline{x} + 5v_2 + \tau \ln \overline{\lambda}_3 &= 0. \end{cases}$$

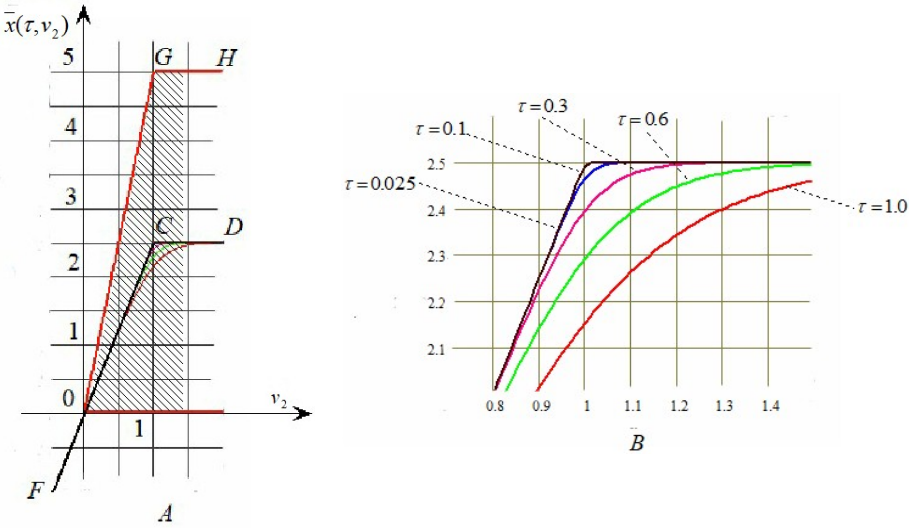



Fig. 2.3.1. Graphs of the function $\overline{x}(\tau, 1, v_2)$ for example 2.3.1

Omitting obvious transformations, we present the final form of the desired approximation for the dependence $x_v^*$ as

$$\overline{x}(\tau, v_1, v_2) = -\tau \ln \left( \sqrt{\frac{v_1^2}{4} + e^{-\frac{5}{\tau}} + e^{-\frac{5v_2}{\tau}}} - \frac{v_1}{2} \right).$$

This function is defined $\forall \tau > 0$, $\forall v_1 \geq 0$ and $\forall v_2 \in \mathbb{R}$. In this domain it has derivatives of any order with respect to the set of all its arguments. The limit of $\overline{x}(\tau, v_1, v_2)$ at $\tau \to +0$ in the case $v_1 > 0$ is $\lim\limits_{\tau \to +0} \overline{x}(\tau, v_1, v_2) = 5v_2$ for $0 \leq v_2 \leq 1$ and $\lim\limits_{\tau \to +0} \overline{x}(\tau, v_1, v_2) = 5$ for $v_2 > 1$.

Now we find the values of the limit $\lim\limits_{\tau \to +0} \overline{x}(\tau, v_1, v_2)$ in cases where $x_v^*$ is not uniquely defined. For $v_1 = 0$ and $0 < v_2 < 1$ we have

$$\lim_{\tau \to +0} \overline{x}(\tau, v_1, v_2) = -\frac{1}{2} \lim_{\tau \to +0} \tau \ln \left( e^{-\frac{5}{\tau}} + e^{-\frac{5v_2}{\tau}} \right) =$$

$$= -\frac{1}{2} \lim_{\tau \to +0} \left( \tau \ln e^{-\frac{5v_2}{\tau}} + \tau \ln \left( 1 + e^{\frac{5v_2 - 5}{\tau}} \right) \right) = \frac{5}{2} v_2 .$$

For $v_1 = 0$ and $v_2 \geq 1$ the result $\lim\limits_{\tau \to +0} \overline{x}(\tau, v_1, v_2) = \frac{5}{2}$ is obtained

similarly:

$$\lim_{\tau\to+0}\overline{x}(\tau,v_1,v_2)=-\frac{1}{2}\lim_{\tau\to+0}\tau\ln\left(e^{-\frac{5}{\tau}}+e^{-\frac{5v_2}{\tau}}\right)=$$

$$=-\frac{1}{2}\lim_{\tau\to+0}\left(\tau\ln e^{-\frac{5}{\tau}}+\tau\ln\left(1+e^{\frac{5-5v_2}{\tau}}\right)\right)=\frac{5}{2}.$$

Next, note that Problem 2.3.1 is inconsistent $\forall v_2<0$ due to the contradictory of the constraints $x\geq 0$ and $x\leq 5v_2$. However, the function $\overline{x}(\tau,v_1,v_2)$ exists $\forall v_1\geq 0$ with $\forall v_2<0$. In this case we have

$$\lim_{\tau\to+0}\overline{x}(\tau,v_1,v_2)=\frac{5}{2}\,v_2\,.$$

This value can be considered as a compromise for conflicting conditions.

In Fig. 1A the broken line $F0GH$ is the graph of the solution $x_v^*$ of Example 2.3.1 $\forall v_1>0$. The broken line $FCD$ is the graph of the same dependency for $v_1=0$. The set of ambiguous values $x_v^*$ is marked with hatching.

Here, the graphs of the approximating function are also shown for the values of the instrumental parameter $\tau=1,\ 0.6,\ 0.3,\ 0.1,\ 0.025$.

Fig. 1B shows on a *larger* scale the same graphs in the vicinity of the point $v_2=1$, where the dependence $x_v^*$ is functional, but not differentiable. The smoothing effect is more noticeable in this figure. The coupling of this effect with the value of $\tau$ is also obvious.

In conclusion, we once again note the important property of approximation using feedback functions of the solution to problem (2.3.1) $x_v^*$.

*The approximating function $\overline{x}(\tau,v_1,v_2)$ turned out to be defined $\forall v_2\in\mathbb{R}$, while the dependence $x_v^*$ itself exists only for $v_2\geq 0$.*

## 2.4. Linear extrapolation in the feedback function method

It follows from Theorem 2.1.2 that for sufficiently small positive values of $\tau$ the feedback function method locally converges to one of the solutions of problem (1.1.1) for $\tau\to+0$. In other words, the approximation error decreases with decreasing absolute value of $\tau$.

However, to obtain a better approximation quality, directly decreasing the modulus of the instrumental parameter $\tau$ does not always lead to the

desired result. The reason for this is the deterioration of the conditionality of the system (2.1.3) at $\tau \to +0$.

Let us consider an alternative algorithm for reducing the error of the method of feedback functions that is free from this drawback.

Continuously differentiable vector functions $\overline{X}(\tau)$ and $\overline{\Lambda}(\tau)$ satisfy the system of equations (2.1.3) on the stationary trajectory of problem (1.1.1). Then the first-order Taylor formulas

$$\overline{X}(\tau+\Delta) = \overline{X}(\tau)+\frac{d\overline{X}}{d\tau}\Delta+o(\Delta) \qquad \text{and} \qquad \overline{\Lambda}(\tau+\Delta) = \overline{\Lambda}(\tau)+\frac{d\overline{\Lambda}}{d\tau}\Delta+o(\Delta)$$

are valid for $\tau \in (0, \tau_0)$ If we pass to the limit at $\Delta \to -\tau + 0$ in these equalities, then the exact solution are

$$X^* = \overline{X}(\tau) - \frac{d\overline{X}}{d\tau}\tau + o(\tau) \qquad \text{and} \qquad \Lambda^* = \overline{\Lambda}(\tau) - \frac{d\overline{\Lambda}}{d\tau}\tau + o(\tau)\ .$$

This gives us the refined estimates for approximating the solutions $X^*$ and $\Lambda^*$ of the form:

$$x^+ = \overline{X}(\tau) - \frac{d\overline{X}}{d\tau}\,\tau \qquad \text{and} \qquad \lambda^+ = \overline{\Lambda}(\tau) - \frac{d\overline{\Lambda}}{d\tau}\,\tau\,. \tag{2.4.1}$$

The values of the derivative components in (2.4.1) are found (according to the theorem on a system of implicit functions) from a system of linear equations

$$\begin{cases} \sum\limits_{j=1}^{n} \dfrac{\partial^2 U}{\partial x_p \partial x_j}\dfrac{d\overline{x}_j}{d\tau} + \sum\limits_{i=1}^{m} \dfrac{\partial^2 U}{\partial x_p \partial \lambda_i}\dfrac{d\overline{\lambda}_i}{d\tau} = -\dfrac{\partial^2 U}{\partial x_p \partial \tau} & p = \overline{1,n}, \\ \sum\limits_{j=1}^{n} \dfrac{\partial^2 U}{\partial \lambda_q \partial x_j}\dfrac{d\overline{x}_j}{d\tau} + \sum\limits_{i=1}^{m} \dfrac{\partial^2 U}{\partial \lambda_q \partial \lambda_i}\dfrac{d\overline{\lambda}_i}{d\tau} = -\dfrac{\partial^2 U}{\partial \lambda_r \partial \tau} & q = \overline{1,m}. \end{cases} \tag{2.4.2}$$

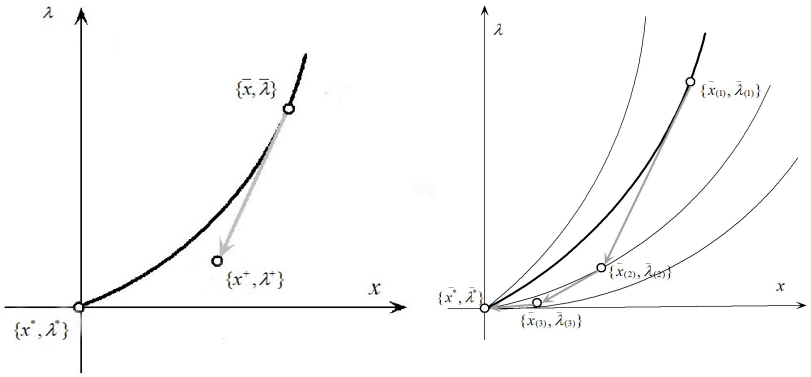
The system is obtained by differentiating the system of equations (2.1.3) with respect to the parameter $\tau$. System (2.4.2) has a non-singular basic matrix (see Theorem 2.1.3) and is similar to the system of equations (2.3.7) for a vector of exogenous parameters $V$.

The following fact is known from the course of mathematical analysis. If the coefficient $\tau$ is sufficiently small in absolute value, then the norm of the difference of the vectors $\{X^+, \Lambda^+\}$ and $\{X^*, \Lambda^*\}$ will be less than the norm of the difference $\{\overline{X}, \overline{\Lambda}\}$ and $\{X^*, \Lambda^*\}$. This is true due to the properties of the function $o(\tau)$.

However, it may turn out that the use of formulas (2.4.1) still does not provide the required level of error. Then the question arises about the possibility and advisability of *iterative* (that is, repeated sequential) use of these formulas.

Formally, it is impossible to apply formulas (2.4.1) to further improve accuracy. Indeed, the point $\{X^+, \lambda^+\}$ is no longer a solution of system (2.1.3) and the implicit function theorem will not be true for it. That is, this point does not belong to the stationary trajectory in the general case (see Fig. 2.4.1A).

The following technique makes it possible to use equalities (2.4.1) for iterative refinement of the solution of problem (1.1.1).




A B

Fig. 2.4.1. Stationary trajectories in sequential linear extrapolation for the method of feedback functions

Let in the statement of problem (1.1.1) the conditions for non-negativity of the components of the vector $X$ be separated into a separate group from the other inequalities $f_i(X) \le 0 \quad i = \overline{1,m}$. Then

$$U(\tau, x, \Lambda) \;=\; L(x, \Lambda) - \sum_{j=1}^{n} R(\tau, x_j) + \sum_{i=1}^{m} R(\tau, \lambda_i)\,. \qquad (2.4.3)$$

We introduce a new instrumental parameter $\vec{\tau}$ with coordinate representation

$$\| \; \tau_{x1} \; \tau_{x2} \; \dots \; \tau_{xn} \; \tau_{\lambda 1} \; \tau_{\lambda 2} \; \dots \; \tau_{\lambda m} \; \|^{\mathrm{T}},$$

whose components are positive constants. Then we replace the scalar parameter $\tau$ with the corresponding component of the vector $\vec{\tau}$ in each term with the function $R$ of formula (2.4.3).

With such a replacement we obtain a family of auxiliary functions

$$U(\vec{\tau}, X, \Lambda) \;=\; L(X, \Lambda) - \sum_{j=1}^{n} R(\tau_{xj}, x_j) + \sum_{i=1}^{m} R(\tau_{\lambda i}, \lambda_i)\,. \qquad (2.4.4)$$

Note that

— firstly, function (2.4.4) coincides with (2.4.3) if all components of vector $\vec{\tau}$ are equal to $\tau$,

— secondly, the basic properties and stationary points of all functions (2.4.4) are the same.

It follows that the family of continuously differentiable auxiliary functions (2.4.4) generates a bundle of smooth stationary trajectories passing through the point $\{X^*, \Lambda^*\}$ (see Fig. 2.4.1в)).

The essence of the proposed method is to select the values of the components of the vector $\vec{\tau}$ that ensure that the refined approximation of the solution $\{X^+, \Lambda^+\}$ falls on some *new* trajectory leading to $\{X^*, \Lambda^*\}$. For this new trajectory, the theorem on implicit functions at the point $\{X^+, \Lambda^+\}$ is satisfied. Therefore, repeated and multiple use of formulas similar to (2.4.1) is already possible.

A graphical interpretation of the described scheme is shown in Fig. 2.4.1в. Since the final trajectory turns out to be a piecewise straight line, the constructed procedure can be called *sequential linear extrapolation.* [4]

The constructed algorithm should also be attributed to instrumental parametric procedures, since additional degrees of freedom are created in it by introducing a *vector* parameter $\vec{\tau}$ of dimension $n + m$.

Now we will give a formal description of the method of linear extrapolation.

Let $t$ be the iteration number. The procedure is performed according to the formulas

$$\begin{cases} \overline{x}_{j(t+1)} \;=\; \overline{x}_{j(t)} - \tau_{xj(t)} \dfrac{d\overline{x}_j}{d\tau_{xj(t)}} & j = \overline{1, n}\,, \\ \overline{\lambda}_{i(t+1)} \;=\; \overline{\lambda}_{i(t)} - \tau_{\lambda i(t)} \dfrac{d\overline{\lambda}_i}{d\tau_{\lambda i(t)}} & i = \overline{1, m}\,. \end{cases} \qquad (2.4.5)$$

These formulas follow from (2.4.1) $\quad \forall t = 0, 1, 2, \ldots$ . The values of the derivatives in (2.4.5) are calculated using formulas similar to (2.4.2) at $\{\overline{X}_{(t)}, \overline{\Lambda}_{(t)}\}$.

[4] It is similar to the procedure considered for the method of smooth penalty functions in [21].

We choose the values $\{\ \tau_{x1}\,\tau_{x2}\,\ldots\,\tau_{xn}\,\tau_{\lambda 1}\,\tau_{\lambda 2}\,\ldots\,\tau_{\lambda m}\ \}$ in formulas (2.4.5) so that the equalities are satisfied

$$\begin{cases} \operatorname*{grad}_x \widehat{U}\Big(\vec{\tau}_{(t+1)}, \overline{X}_{(t+1)}, \overline{\Lambda}_{(t+1)}\Big) = o\,, \\ \operatorname*{grad}_\lambda \widehat{U}\Big(\vec{\tau}_{(t+1)}, \overline{X}_{(t+1)}, \overline{\Lambda}_{(t+1)}\Big) = o\,. \end{cases} \tag{2.4.6}$$

In our case, by virtue of (2.1.1), this recalculation is reduced to solving $n+m$ *linear equations with one unknown* each, having the form

$$\begin{cases} \dfrac{\partial L}{\partial x_j}\big(\,\overline{X}_{(t)}, \overline{\Lambda}_{(t)}\big) - \tau_{xj(t+1)}\Psi(\overline{x}_{j(t)}) = 0 \qquad j = \overline{1,n}\,, \\ \dfrac{\partial L}{\partial \lambda_i}\big(\,\overline{X}_{(t)}, \overline{\Lambda}_{(t)}\big) - \tau_{\lambda i(t+1)}\Psi(\overline{\lambda}_{i(t)}) = 0 \qquad i = \overline{1,m} \end{cases}$$

or

$$\begin{cases} \dfrac{\partial L}{\partial x_j}\big(\,\overline{X}_{(t)}, \overline{\Lambda}_{(t)}\big) - \tau_{xj(t+1)}\Psi(\overline{x}_{j(t)}) = 0 \qquad j = \overline{1,n}\,, \\ f_i\left(\,\overline{X}_{(t)}\,\right) - \tau_{\lambda i(t+1)}\Psi(\overline{\lambda}_{i(t)}) = 0 \qquad\qquad i = \overline{1,m}\,. \end{cases} \tag{2.4.7}$$

Finally, the values of the derivatives in (2.4.5) for each iteration $t$ are found from the system

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \frac{\partial^2 U}{\partial x_p \partial x_j}\frac{d\overline{x}_j}{d\tau_{xj(t)}} + \sum_{i=1}^{m} \frac{\partial^2 U}{\partial x_p \partial \lambda_i}\frac{d\overline{\lambda}_i}{d\tau_{\lambda i(t)}} = -\frac{\partial^2 U}{\partial x_p \partial \tau_{xp(t)}} \qquad p = \overline{1,n}, \\ \displaystyle\sum_{j=1}^{n} \frac{\partial^2 U}{\partial \lambda_q \partial x_j}\frac{d\overline{x}_j}{d\tau_{xj(t)}} + \sum_{i=1}^{m} \frac{\partial^2 U}{\partial \lambda_q \partial \lambda_i}\frac{d\overline{\lambda}_i}{d\tau_{\lambda i(t)}} = -\frac{\partial^2 U}{\partial \lambda_q \partial \tau_{\lambda q(t)}} \qquad q = \overline{1,m}, \end{cases} \tag{2.4.8}$$

which is similar to (2.4.2).

The iterative application of formulas (2.4.5)—(2.4.8) is a locally convergent procedure. This convergence follows from the properties of the Taylor approximation and the obvious compressibility of the operator defined by formulas (2.4.5)—(2.4.8) when the norm of the vector $\vec{\tau}$ tends to zero.

The practical application of the successive linear extrapolation procedure is illustrated by

`Example 2.4.1.`

*Maximize* $F = -(x_1-1)^2 - x_2^2$

*subject to* $x_1 \ge 0\,,\ \ x_2 \ge 0$ *and* $x_1 + 2x_2 \le 3\,,\ \ x_1^2 - x_2 \le 0\,.$

`Solution.`

Let the feedback function $Q(\tau, s) = \dfrac{\tau}{2}\left(s - \dfrac{1}{s}\right)$, then the corresponding function $R(\tau, s) = \dfrac{\tau}{2}\left(\dfrac{s^2}{2} - \ln s - \dfrac{1}{2}\right)$.

Let us introduce the notation

$$\begin{aligned} f_1(x) &= -3 + x_1 + 2x_2\,, \\ f_2(x) &= x_1^2 - x_2\,, \\ L(x,\lambda) &= -(x_1-1)^2 - x_2^2 - \lambda_1(-3 + x_1 + 2x_2) - \lambda_2(x_1^2 - x_2)\,. \end{aligned}$$

In this example, instead of the scalar parameter $\tau$ we use a four-dimensional vector $\vec{\tau}$ with a coordinate column $\| \tau_{x1}\ \tau_{x2}\ \tau_{\lambda 1}\ \tau_{\lambda 2} \|^{\mathrm{T}}$. The auxiliary $U$-function has the form

$$\begin{aligned} U(\tau, x, \lambda) = -(x_1-1)^2 - x_2^2 - \lambda_1(-3 + x_1 + 2x_2) - \lambda_2(x_1^2 - x_2) - \\ -R(\tau_{x1}, x_1) - R(\tau_{x2}, x_2) + R(\tau_{\lambda 1}, \lambda_1) + R(\tau_{\lambda 2}, \lambda_2)\,. \end{aligned}$$

Then the conditions for the $U$-function stationarity for the problem being solved, that is, the system of equations (2.1.4) will be

$$\left\{ \begin{aligned} -3 + \overline{x}_1 + 2\overline{x}_2 &= \frac{\tau}{2}\left(\overline{\lambda}_1 - \frac{1}{\overline{\lambda}_1}\right), \\ \overline{x}_1^2 - \overline{x}_2 &= \frac{\tau}{2}\left(\overline{\lambda}_2 - \frac{1}{\overline{\lambda}_2}\right), \\ -2\left(\overline{x}_1 - 1\right) - \overline{\lambda}_1 - 2\overline{x}_1\overline{\lambda}_2 &= \frac{\tau}{2}\left(\overline{x}_1 - \frac{1}{\overline{x}_1}\right), \\ -2\overline{x}_2 - 2\overline{\lambda}_1 + \overline{\lambda}_2 &= \frac{\tau}{2}\left(\overline{x}_2 - \frac{1}{\overline{x}_2}\right). \end{aligned} \right. \tag{2.4.9}$$

We will take the solution of system (2.4.9) with $\tau = 0.01$, as an initial approximation in the procedure of successive linear extrapolation. This solution is given in tables 2.4.1a—2.4.1b.

T a b l e 2.4.1a.
Solutions of system (2.4.9) for $\tau = 0.01$

| $\tau$ | $\overline{x}_1(\tau)$ | $\overline{x}_2(\tau)$ | $\overline{\lambda}_1(\tau)$ | $\overline{\lambda}_2(\tau)$ | $F(\overline{x}(\tau))$ |
|---|---|---|---|---|---|
| $10^{-2}$ | 0.59002481 | 0.35181724 | $2.930*10^{-3}$ | 0.69704208 | -0.51620893 |

T a b l e 2.4.1b.
Solutions of system (2.4.9) for $\tau = 0.01$

| $f_1(\overline{x}(\tau))$ | $f_2(\overline{x}(\tau))$ | $L(\overline{x}, \overline{\lambda})$ | $U(\overline{x}, \overline{\lambda})$ |
|---|---|---|---|
| -1.70634071 | $-3.688*10^{-3}$ | -0.50863832 | -0.48549662 |

We will perform a sequential refinement of the values of primary and dual variables according to formulas (2.4.5) using solutions of the system of linear equations (2.4.8).

The main matrix of (2.4.8) here has the form

$$\begin{Vmatrix} -\frac{\tau}{2}\left(1+\frac{1}{\overline{x}_1^2}\right)-2-2\overline{\lambda}_2 & 0 & -1 & -2\overline{x}_1 \\ 0 & -\frac{\tau}{2}\left(1+\frac{1}{\overline{x}_2^2}\right)-2 & -2 & 1 \\ -1 & -2 & \frac{\tau}{2}\left(1+\frac{1}{\overline{\lambda}_1^2}\right) & 0 \\ -2x_1 & 1 & 0 & \frac{\tau}{2}\left(1+\frac{1}{\overline{\lambda}_2^2}\right) \end{Vmatrix}.$$

The column of right-hand sides of this system will be

$$\left\| \; -\frac{\tau}{2}\left(\overline{x}_1 - \frac{1}{\overline{x}_1}\right) \quad -\frac{\tau}{2}\left(\overline{x}_2 - \frac{1}{\overline{x}_2}\right) \quad \frac{\tau}{2}\left(\overline{\lambda}_1 - \frac{1}{\overline{\lambda}_1}\right) \quad \frac{\tau}{2}\left(\overline{\lambda}_2 - \frac{1}{\overline{\lambda}_2}\right) \; \right\|^{\mathrm{T}}.$$

Let in procedure (2.4.5) the refined values of the variables at iteration number $t-1 \quad \forall t = 1, 2, \ldots$ be equal, respectively, to

$$\overline{x}_{1(t)},\, \overline{x}_{2(t)}$$

and

$$\overline{\lambda}_{1(t)},\, \overline{\lambda}_{2(t)}\,.$$

Then the values of the components $\vec{\tau}_{(t)}$, guaranteeing the belonging of the point $\{\overline{X}_{(t)}, \overline{\Lambda}_{(t)}\}$ to some other stationary trajectory, by (2.4.7) will be determined by the equalities

$$\begin{cases} \tau_{x1(t)} &= \dfrac{2\Big(-2(\overline{x}_{1(t)} - 1) - \overline{\lambda}_{1(t)} - 2\overline{x}_{1(t)}\overline{\lambda}_{2(t)}\Big)}{\overline{x}_{1(t)} - \dfrac{1}{\overline{x}_{1(t)}}}, \\ \tau_{x2(t)} &= \dfrac{2\Big(-2\overline{x}_{2(t)} - 2\overline{\lambda}_{1(t)} + \overline{\lambda}_{2(t)}\Big)}{\overline{x}_{2(t)} - \dfrac{1}{\overline{x}_{2(t)}}}, \\ \tau_{\lambda 1(t)} &= \dfrac{2\Big(-3 + \overline{x}_{1(t)} + 2\overline{x}_{2(t)}\Big)}{\overline{\lambda}_{1(t)} - \dfrac{1}{\overline{\lambda}_{1(t)}}}, \\ \tau_{\lambda 2(t)} &= \dfrac{2\Big(\overline{x}^2_{1(t)} - \overline{x}_{2(t)}\Big)}{\overline{\lambda}_{2(t)} - \dfrac{1}{\overline{\lambda}_{2(t)}}}. \end{cases}$$

The value of the approximation error can be estimated in different ways.

Firstly, by the values of the dual variables $\lambda_1$ and $\lambda_2$, the exact values of which in this problem are found analytically. They are equal to 0 and $\lambda_2^*$, respectively, where $\lambda_2^*$ is the positive root of the equation

$$\lambda_2(\lambda_2+1)^2-2=0\,.$$

This root is (according to Cardano's formula)

$$\lambda_2^*=\frac{\left(\sqrt[3]{28+3\sqrt{87}}-1\right)^2}{3\sqrt[3]{28+3\sqrt{87}}}\approx 0.69562077\,.$$

Secondly, this can be done by comparing the values of the objective function $F$ and the Lagrange function $L$.

The results of the calculations performed during the solution of example 2.4.1 are given in tables 2.4.2, 2.4.3 and 2.4.4.

T a b l e 2.4.2

| $t$ | $\overline{x}_{1(t)}$ | $\overline{x}_{2(t)}$ | $F$ | $\overline{\lambda}_{1(t)}$ | $\overline{\lambda}_{2(t)}$ |
|---|---|---|---|---|---|
| 0 | 0.590024813 | 0.351817238 | −0.291855023 | 0.002930222 | 0.697042081 |
| 1 | 0.589788382 | 0.347873253 | −0.289289372 | $-1.390\cdot 10^{-5}$ | 0.695539663 |
| 2 | 0.589754522 | 0.347810387 | −0.289289372 | $1.294\cdot 10^{-9}$ | 0.695620736 |
| 3 | 0.589754512 | 0.347810385 | −0.289273424 | 0 | 0.695620770 |

T a b l e 2.4.3

| $t$ | $f_1$ | $f_2$ | $L$ | $L-F$ | $U$ |
|---|---|---|---|---|---|
| 0 | −1.706340711 | −0.003687958 | −0.284284404 | 0.007570619 | −0.261142713 |
| 1 | −1.714465112 | $-2.291\cdot 10^{-5}$ | −0.289297265 | $-7.893\cdot 10^{-6}$ | −0.289593289 |
| 2 | −1.714624703 | $8.824\cdot 10^{-9}$ | −0.289273422 | $-3.920\cdot 10^{-9}$ | −0.289273422 |
| 3 | −1.714624718 | $-1.11\cdot 10^{-15}$ | −0.289273424 | 0 | −0.290232652 |

Table 2.4.4

| $t$ | $\tau_{x1(t)}$ | $\tau_{x2(t)}$ | $\tau_{\lambda 1(t)}$ | $\tau_{\lambda 2(t)}$ | $\overline{\lambda}_{2(t)} - \lambda_2^*$ |
|---|---|---|---|---|---|
| 0 | 0.01 | 0.01 | 0.01 | 0.01 | 0.001421311 |
| 1 | $9.565 \cdot 10^{-6}$ | 0.000141717 | $-4.767 \cdot 10^{-5}$ | $6.176 \cdot 10^{-5}$ | $-8.111 \cdot 10^{-5}$ |
| 2 | $-8.794 \cdot 10^{-9}$ | $3.265 \cdot 10^{-8}$ | $4.437 \cdot 10^{-9}$ | $-2.379 \cdot 10^{-8}$ | $-3.373 \cdot 10^{-8}$ |
| 3 | $-1.41 \cdot 10^{-15}$ | 0 | 0 | $2.99 \cdot 10^{-15}$ | $-4.40 \cdot 10^{-10}$ |

# Chapter 3.

# Feedback functions in extremum search problems

## 3.1. Global extremum of functions

The method of feedback functions allows us to solve not only the problems (1.1.1) or (2.3.1). We can also solve various optimization problems reduced to them.

Let us first consider the use of this method for a simple search for an extremum in a finite set of numbers.

Let $f^*$ also be value of the maximal element within $\{v_1, v_2, \ldots, v_K\}$. It is easy to check, that $f^*$ is the optimal value of the objective function in the linear programming problem:

$$\textit{minimize} \qquad f \qquad \textit{subject to:} \qquad f \geq v_i \quad i = \overline{1, K}.$$

or

$$\textit{maximize} \qquad -f \qquad \textit{subject to:} \quad v_i - f \leq 0 \quad i = \overline{1, K}. \tag{3.1.1}$$

The variable $f$ has no sign restriction, therefore the auxiliary function (2.1.2) is

$$U(\tau, f, \Lambda, V) = -f - \sum_{i=1}^{K} \lambda_i (v_i - f) + \sum_{i=1}^{K} R(\tau, \lambda_i)\,, \tag{3.1.2}$$

when the problem (3.1.1) is solved by the method of feedback functions.

If we choose $Q(\tau, s) = \tau \ln s$, then the stationarity condition of function (3.1.2) (that is, the system of equations (2.1.3)) takes the form

$$\left\{ \begin{array}{rcll} -1 + \sum\limits_{i=1}^{K} \overline{\lambda}_i & = & 0, & \\ -v_i + \overline{f} + \tau \ln \overline{\lambda}_i & = & 0 & i = \overline{1, K} \end{array} \right. \quad \Longrightarrow$$

$$\Longrightarrow \quad \left\{ \begin{array}{rcll} \overline{\lambda}_i & = & e^{\frac{v_i - \overline{f}}{\tau}} & i = \overline{1, K}\,, \\ \overline{f} & = & \tau \ln \left( \sum\limits_{i=1}^{K} e^{\frac{v_i}{\tau}} \right). & \end{array} \right. \tag{3.1.3}$$

From where we obtain the values of the maximum and, respectively, the minimum number in the set $\left\{ v_1, v_2, \ldots, v_K \right\}$

$$f^*_{\max} = \lim_{\tau \to +0} \tau \ln \left( \sum_{i=1}^{K} e^{\frac{v_i}{\tau}} \right) \quad \text{and} \quad f^*_{\min} = - \lim_{\tau \to +0} \tau \ln \left( \sum_{i=1}^{K} e^{-\frac{v_i}{\tau}} \right). \tag{3.1.4}$$

Let us now estimate the error of the value $f^*_{\max}$ obtained by the formula $\overline{f}(\tau) = \tau \ln \left( \sum\limits_{i=1}^{K} e^{\frac{v_i}{\tau}} \right)$. We will assume that the numbers of set $\left\{ v_1, v_2, \ldots, v_K \right\}$ are arranged in descending order. Let the the first $M$ (here $M < K$) of them are equal to $f^*$. Then we have

$$\overline{f}(\tau) - f^* = \tau \ln \left( \sum_{i=1}^{K} e^{\frac{v_i}{\tau}} \right) - \tau \ln e^{\frac{f^*}{\tau}} = \tau \ln \left( M + \sum_{i=M+1}^{K} e^{\frac{v_i - f^*}{\tau}} \right).$$

Here, on the one hand,

$$\overline{f}(\tau) - f^* \le \tau \ln \left( M + (K - M) e^{\frac{A - f^*}{\tau}} \right) =$$

$$= \tau \ln M + \tau \ln \left( 1 + \frac{K - M}{M} e^{\frac{A - f^*}{\tau}} \right) \;\le\; \tau \ln M + \tau e^{\frac{A - f^*}{\tau}},$$

where $A = v_{M+1}$. On the other hand, it is obvious $\overline{f}(\tau) - f^* \ge \tau \ln M$.

These inequalities mean that the main part of the error at $\tau \to +0$ will be $\tau \ln M$, when $M \ge 2$. That is, in the case when in the studied set the

maximum number is not unique. If $M = 1$, then the upper estimate of the error will be determined by the value $\tau e^{\frac{A-f^*}{\tau}}$.

Note that this estimate of the error is obtained due to the special form of the feedback function. If, for example, we take

$$Q(\tau, s) = \frac{\tau}{2}\left(s - \frac{1}{s}\right) \qquad \text{c} \qquad R(\tau, s) = \frac{\tau}{2}\left(\frac{s^2}{2} - \ln s + \frac{1}{2}\right),$$

then the system of equations (3.1.3) takes the form

$$\left\{ \begin{array}{rcl} -1 + \sum\limits_{i=1}^{K} \overline{\lambda}_i & = & 0, \\ v_i - \overline{f} & = & \dfrac{\tau}{2}\left(\overline{\lambda}_i - \dfrac{1}{\overline{\lambda}_i}\right) \qquad i = \overline{1, K}\,, \end{array} \right. \tag{3.1.5}$$

for which only a numerical solution is possible.

For illustration, Table 3.1.1 presents the results of solving system (3.1.3) for a set of numbers $\{\ 5,\ -2,\ 4,\ 7,\ 0\ \}$ for various values of the parameter $\tau$.

T a b l e 3.1.1.

| $\tau$ | $\overline{f}(\tau)$ | $\overline{\lambda}_1(\tau)$ | $\overline{\lambda}_2(\tau)$ | $\overline{\lambda}_3(\tau)$ | $\overline{\lambda}_4(\tau)$ | $\overline{\lambda}_5(\tau)$ |
|---|---|---|---|---|---|---|
| $10^{-0.00}$ | 7.170719212 | 0.114095529 | $1.0.0^{-4}$ | 0.041973399 | 0.843058261 | $7.6877 * 10^{-}$ |
| $10^{-0.25}$ | 7.018454440 | 0.027615558 | $1.0841 * 10^{-7}$ | $4.6651 * 10^{-3}$ | 0.967715479 | $3.7990 * 10^{-}$ |
| $10^{-0.50}$ | 7.000590038 | $1.7884 * 10^{-3}$ | $4.355 * 10^{-13}$ | $7.5703 * 10^{-5}$ | 0.998135874 | $2.430 * 10^{-1}$ |
| $10^{-0.75}$ | 7.000002329 | $1.3048 * 10^{-5}$ | $4.355 * 10^{-13}$ | $4.7135 * 10^{-8}$ | 0.999986904 | 0.000000000 |
| $10^{-1.00}$ | 7.000000000 | $2.0612 * 10^{-9}$ | 0.000000000 | $9.358 * 10^{-14}$ | 0.999999998 | 0.000000000 |
| $10^{-1.25}$ | 7.000000000 | 0.000000000 | 0.000000000 | 0.000000000 | 1.000000000 | 0.000000000 |

In table 3.1.2 for comparison, the numerical solutions of system (3.1.5) for the set of numbers $\{\,5,\,5,\,4,\,5,\,0\,\}$ also for different values of the parameter $\tau$.

T a b l e 3.1.2.

| $\tau$ | $\overline{f}(\tau)$ | $\overline{\lambda}_1(\tau)$ | $\overline{\lambda}_2(\tau)$ | $\overline{\lambda}_3(\tau)$ | $\overline{\lambda}_4(\tau)$ | $\overline{\lambda}_5(\tau)$ |
|---|---|---|---|---|---|---|
| $10^{-1.00}$ | 5.109862742 | 0.333328289 | 0.333328289 | $1.5133 * 10^{-5}$ | 0.333328289 | 0.000000000 |
| $10^{-1.20}$ | 5.069317752 | 0.333333319 | 0.333333319 | $4.3629 * 10^{-8}$ | 0.333333319 | 0.000000000 |
| $10^{-1.50}$ | 5.034741173 | 0.333333333 | 0.333333333 | 0.000000000 | 0.333333333 | 0.000000000 |
| $10^{-2.00}$ | 5.010986124 | 0.333333333 | 0.333333333 | 0.000000000 | 0.333333333 | 0.000000000 |
| $10^{-4.00}$ | 5.000109861 | 0.333333333 | 0.333333333 | 0.000000000 | 0.333333333 | 0.000000000 |
| $10^{-7.00}$ | 5.000000110 | 0.333333333 | 0.333333333 | 0.000000000 | 0.333333333 | 0.000000000 |

We now consider the problem of finding extreme values for numerical sets of cardinality *continuum*.

Let the function $f(x)$ be integrable on a compact set with non-empty interior $\Omega \subset E^n$. Replacing the summation operation with integration in formulas (3.1.4), we obtain an estimate of the value of the global maximum of a function of several variables of the form:

$$f^*_{\max} = \lim_{\tau \to +0} \tau \ln \int\limits_{\Omega} e^{\frac{f(x)}{\tau}}\, dx\,. \tag{3.1.6}$$

Justice (3.1.6) follows from conditions $f(x) \le f^*$ and estimates:

$$\lim_{\tau \to +0} \tau \ln \int\limits_{\Omega} e^{\frac{f(x)}{\tau}}\, dx = \lim_{\tau \to +0} \tau \ln e^{\frac{f^*}{\tau}} \int\limits_{\Omega} e^{\frac{f(x)-f^*}{\tau}}\, dx =$$

$$= f^* + \lim_{\tau \to +0} \tau \ln \int\limits_{\Omega} e^{\frac{f(x)-f^*}{\tau}}\, dx\,,$$

$$0 \le \lim_{\tau \to +0} \tau \ln \int\limits_{\Omega} e^{\frac{f(x)-f^*}{\tau}}\, dx \le \lim_{\tau \to +0} \tau \ln \operatorname{mes} \Omega = 0\,.$$

Note that the connection between the operations of integration and searching for an extremum is used to solve problems of different classes. For example, when searching for maximin in problems of *game theory* [27], or when calculating integrals *by the stationary-point method* [28].

The following examples illustrate the application of formula (3.1.6).

`Example 3.1.1.` *Find on the set* $\Omega$ *the global extrema for the function* $f(x) = |x-1| - 2|x|$ *by* $x \in E^1$. *The set* $\Omega = \{-1 \le x \le v\}$ *depends on the exogenous parameter* $v \in [1,3]$.

`Solution.` For the global maximum $\forall v \in [1,3]$ we have

$$f^*_{\max} = \lim_{\tau \to +0} \tau \ln \int_{-1}^{v} e^{\frac{|x-1|-2|x|}{\tau}}\, dx =$$

$$= \lim_{\tau \to +0} \tau \ln \left[ \int_{-1}^{0} e^{\frac{x+1}{\tau}}\, dx + \int_{0}^{1} e^{\frac{-3x+1}{\tau}}\, dx + \int_{1}^{v} e^{\frac{-x-1}{\tau}}\, dx \right] =$$

$$= \lim_{\tau \to +0} \tau \ln \tau \left[ \left( e^{\frac{1}{\tau}} - 1 \right) + \frac{1}{3} \left( e^{\frac{1}{\tau}} - e^{-\frac{2}{\tau}} \right) + \left( e^{-\frac{2}{\tau}} - e^{-\frac{v+1}{\tau}} \right) \right] = 1\,.$$

It is achieved at $x = 0 \quad \forall v \in [1,3]$.

For the global minimum we have

$$f^*_{\min} = - \lim_{\tau \to +0} \tau \ln \int_{-1}^{v} e^{-\frac{|x-1|-2|x|}{\tau}}\, dx =$$

$$= - \lim_{\tau \to +0} \tau \ln \left[ \int_{-1}^{0} e^{-\frac{x+1}{\tau}}\, dx + \int_{0}^{1} e^{\frac{3x-1}{\tau}}\, dx + \int_{1}^{v} e^{\frac{x+1}{\tau}}\, dx \right] =$$

$$= - \lim_{\tau \to +0} \tau \ln \left[ \tau \left( 1 - e^{-\frac{1}{\tau}} \right) + \frac{\tau}{3} \left( e^{\frac{2}{\tau}} - e^{-\frac{1}{\tau}} \right) + \tau \left( e^{\frac{v+1}{\tau}} - e^{\frac{2}{\tau}} \right) \right] = -v - 1.$$

This minimum is reached at $x = v \quad \forall v \in [1,3]$.

The graph of the function $y = |x-1| - 2|x|$ is shown in Fig. 3.1.1. The set $\Omega$ is highlighted in gray. Point $A$ is the global maximum, point $B$ is the global minimum.

`Example 3.1.2.` *Find global extrema by* $x \in \Omega \subseteq E^2$ *of the function* $f(x) = |x_1| + |x_2 - 1|$, *where*

$$\Omega = \left\{ \begin{array}{l} -1 \le x_1 \le 1, \\ 0 \le x_2, \\ 2x_1 + x_2 - 2 \le 0. \end{array} \right\}.$$

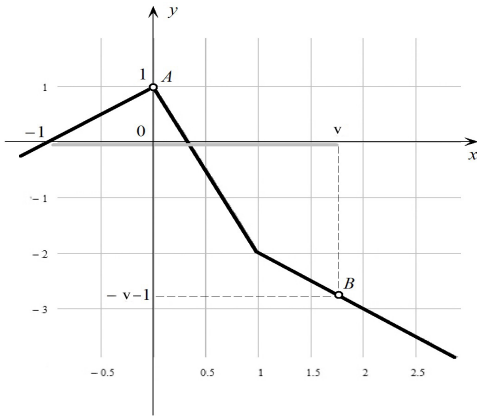



Fig. 3.1.1. Graph of the function $y = |x-1| - 2|x|$.

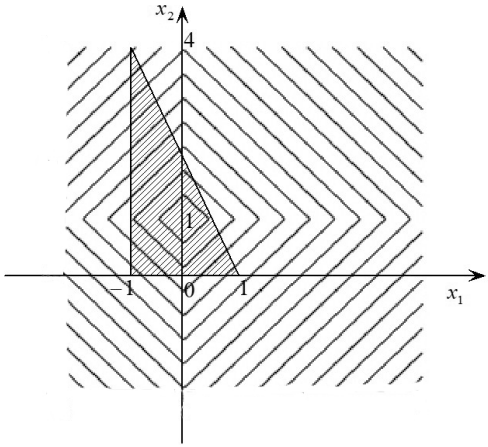



Fig. 3.1.2. System of isolines for $f(x) = |x_1| + |x_2 - 1|$.

`Solution.` For the global maximum we have:

$$f^*_{\max} = \lim_{\tau \to +0} \tau \ln \iint\limits_{\Omega} e^{\frac{|x_1|+|x_2-1|}{\tau}}\, dx_1 dx_2 =$$

$$= \lim_{\tau \to +0} \tau \ln \frac{\tau^2}{3} e^{\frac{2}{\tau}} \left(5 - 10e^{-\frac{1}{\tau}} + e^{\frac{2}{\tau}} - 8e^{-\frac{3}{2\tau}} + 12e^{-\frac{2}{\tau}}\right) = 4\,.$$

It is reached at the boundary of the domain $\Omega$ at $x_1 = -1$ and $x_2 = 4$ (see Fig. 3.1.2).

The global minimum is internal
non-smooth, at the point $x_1 = 0$, $x_2 = 1$

$$f^*_{\min} = -\lim_{\tau \to +0} \tau \ln \iint\limits_{\Omega} e^{-\frac{|x_1|+|x_2-1|}{\tau}}\, dx_1 dx_2 =$$

$$= -\lim_{\tau \to +0} \tau \ln \frac{\tau^2}{3} \left(5e^{-\frac{2}{\tau}} - 10e^{-\frac{1}{\tau}} + e^{-\frac{4}{\tau}} - 8e^{-\frac{1}{2\tau}} + 12\right) = 0.$$

Note also that the optimal value of the objective function in the mathematical programming problem

$$\begin{array}{lll} \textit{maximize} & F(x) & x \in E^n \\ \textit{subject to} & & x \in \Omega \subseteq E^n \end{array}$$

can be represented (under appropriate assumptions on the properties of $F(x)$ and $\Omega$) as

$$F^* = \lim_{\tau \to +0} \tau \ln \int\limits_{\Omega} e^{\frac{F(x)}{\tau}} \, dx \, .$$

## 3.2. Maximum function

The feedback function method may also be useful in solving problems more complex than (2.3.1).

Let us consider problems in the formulation of which superpositions of extremum search operations are used[1]

Consider the following problem:

*find the minimum with respect to* $X \quad Y(X)$:
*provided* $X \in \Omega$, *where* $\Omega \subseteq E^n$ *is compact, and*

$$Y(X) = \max_{k=\overline{1,K}} \{y_k(X)\} \, , \tag{3.2.1}$$

assuming that the functions $y_k(X) \quad k = \overline{1,K}$, are continuously differentiable on $\Omega$.

In the case when the set of admissible arguments $\Omega$ is defined by a system of inequalities of the form $f_i(X) \leq 0 \quad i = \overline{1,m}$, the problem (3.2.1) is equivalent to the mathematical programming problem:

*maximize with respect to* $\{X, v\}$ *the function* $-v$
*subject to*

$$y_k(X) - v \leq 0 \quad k = \overline{1,K} \tag{3.2.2}$$

And

$$f_i(X) \leq 0 \qquad i = \overline{1,m}.$$

Here $v$ is an independent scalar variable. We also assume that the functions $f_i(X) \quad i = \overline{1,m}$, are continuously differentiable on the set $\Omega$.

[1] See, for example, [2], [14]. An alternative method of such analysis, based on the use of external type penalty functions, was proposed in [11]. It was substantiated and developed in [27].

The dependence $v_x^*$ under the assumptions made is a continuous, but non-differentiable function in its entire domain of definition. Therefore, to solve problem (3.2.2), we apply the method of feedback functions with an auxiliary function

$$U(\tau, X, \Lambda, v) = -v - \sum_{k=1}^{K} \mu_k \Big( y_k(X) - v \Big) - \sum_{i=1}^{m} \lambda_i f_i(X) + \\ + \sum_{k=1}^{K} R\big(\tau, \mu_k\big) + \sum_{k=1}^{m} R\big(\tau, \lambda_i\big), \qquad (3.2.3)$$

where $\Lambda = \{\mu_1, \ldots, \mu_K, \lambda_1, \ldots, \lambda_m, \}$

The stationarity conditions for function (3.2.3) can be written as a system of equations:

$$\begin{cases} \sum\limits_{k=1}^{K} \overline{\mu}_k \cdot \operatorname*{grad}_x y_k(\overline{X}) + \sum\limits_{i=1}^{m} \overline{\lambda}_i \cdot \operatorname*{grad}_x f_i(\overline{X}) &=& o\,, \\ -y_k(\overline{X}) + \overline{v} + Q\Big(\tau, \overline{\mu}_k\Big) &=& 0 \qquad k = \overline{1, K}\,, \\ -f_i(\overline{X}) + Q\Big(\tau, \overline{\lambda}_i\Big) &=& 0 \qquad i = \overline{1, m}\,, \\ -1 + \sum\limits_{k=1}^{K} \overline{\mu}_k &=& 0\,. \end{cases} \qquad (3.2.4)$$

If the minimax problem has no constraints, and the feedback function is defined as $Q(\tau, s) = \tau \ln s$, then the stationarity conditions of the auxiliary function are simplified

$$\begin{cases} \sum\limits_{k=1}^{K} \overline{\lambda}_k \cdot \operatorname*{grad}_x y_k(\overline{X}) &=& o\,, \\ \overline{\mu}_k &=& e^{\frac{y_k(\overline{X}) - \overline{v}}{\tau}} \qquad k = \overline{1, K}\,, \\ -1 + \sum\limits_{k=1}^{K} \overline{\mu}_k = 0\,. \end{cases} \qquad (3.2.5)$$

Note that from the last two equalities a smoothed approximation of the function under study is obtained $\overline{v}(\tau, \overline{X}) = \tau \ln \sum\limits_{k=1}^{K} e^{\frac{y_k(\overline{X})}{\tau}}$. The first two equalities in (3.2.5) are necessary conditions for stationarity in $X$ for this approximation.

As an illustration, consider

`Example 3.2.1.` *Find the minimum of* $v(x) = \max\left\{x^2\,;\ \sin 4x\right\}.$

`Solution.` This problem is reduced to a mathematical programming problem

$$\begin{array}{ll} \textit{maximize by } \{x, v\} & -v \\ \textit{subject to:} & x^2 - v \le 0\,, \qquad \sin 4x - v \le 0\,. \end{array}$$

From (3.2.5) we obtain that the approximation of the maximum function has the form

$$\overline{v}(x) = \tau \ln\left(e^{\frac{x^2}{\tau}} + e^{\frac{\sin 4x}{\tau}}\right). \tag{3.2.6}$$

Its stationary points are the roots of the equation

$$\overline{x} e^{\frac{\overline{x}^2}{\tau}} + 2e^{\frac{\sin 4\overline{x}}{\tau}} \cos 4\overline{x} = 0\,. \tag{3.2.7}$$

Equation (3.2.7) has three roots, which are approximations to the points

$$x_1^* = 0\,, \quad x_2^* = \frac{\pi}{8} = 0.392699082 \quad \text{and} \quad x_3^* = 0.669283188\,,$$

the last of which is a positive root of the equation $x^2 = \sin 4x$.

Solutions of equation (3.2.7) for different values of the parameter $\tau$ are given in Table 3.2.1, and the graphs of the function $\overline{v}(x)$ are shown in Fig. 3.2.1.

T a b l e 3.2.1

| $\tau$ | $\overline{x}_1(\tau)$ | $\overline{x}_2(\tau)$ | $\overline{x}_3(\tau)$ |
|---|---|---|---|
| 0.50 | −0.166596357 | 0.413732397 | 0.723794952 |
| 0.25 | −0.124388280 | 0.398475953 | 0.704617207 |
| 0.10 | −0.077449382 | 0.392845484 | 0.688202392 |
| 0.05 | −0.046509302 | 0.392699415 | 0.679216477 |
| *Точное решение* | 0 | 0.392699082 | 0.669283188 |

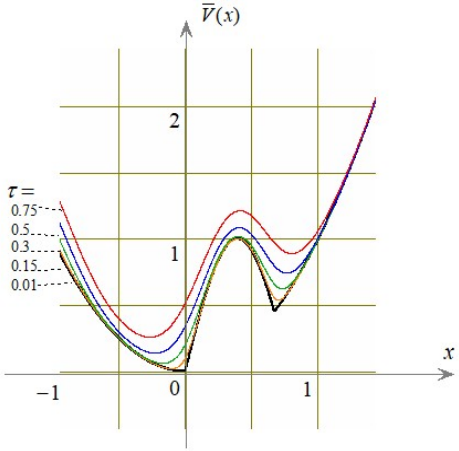



Fig. 3.2.1. Graphical interpretation of problem 3.2.1

As another illustration, let us consider a problem that arises in the process of searching by the dichotomy method for an extremum of a function of $n$ variables in a given direction in $E^n$.

Let a continuous, unimodal (that is, having a single extremum) on the segment $[\alpha, \beta]$ function of one variable $f(x)$ be given. It is required to find on this segment two points, the values of $f(x)$ at which, allow us to construct a new segment of the minimum possible length, containing the extremum point.

Let us denote the required points as $x_1$ and $x_2$. Then from the properties of unimodality it follows that these points are the solution of the following problem:

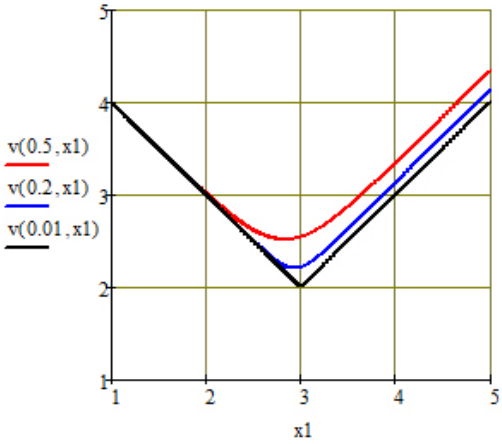
`Example 3.2.2.` *Find the minimum by* $\{x_1, x_2\}$ *of* $\max\{x_2-\alpha, \beta-x_1\}$ *provided* $x_1 \le x_2$.

`Solution.` This minimax problem reduces to a parametric programming problem of the form

$$\begin{array}{llll} & \textit{find the maximum by } \{v, x_1, x_2\} & & -v \\ & \textit{under conditions} & & \\ x_2 & - \alpha & \le & v\,, \\ \beta & - x_1 & \le & v\,, \\ x_1 & - x_2 & \le & 0\,. \end{array}$$

An auxiliary function for this problem will be

$$U(\tau, X, \Lambda, V) = -v - \lambda_1(x_2 - \alpha - v) - \lambda_2(\beta - x_1 - v) - \\ -\lambda_3(x_1 - x_2) + R(\tau, \lambda_1) + R(\tau, \lambda_2) + R(\tau, \lambda_3)\,.$$




Rice. 3.2.2. Graphical representation of coordinated approximations for problem 3.2.2

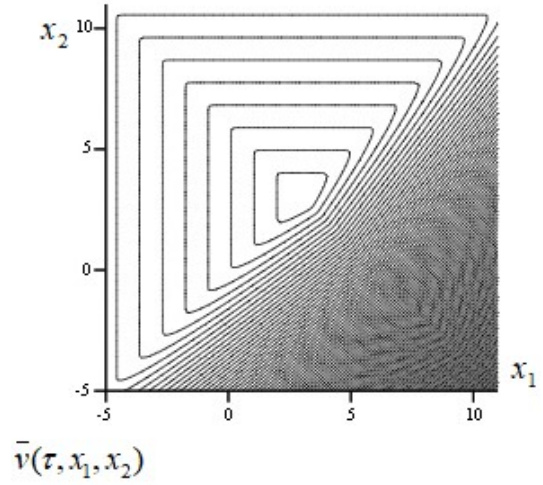



Fig. 3.2.3. Graphical representation of the system of isolines of the function $\overline{v}(\tau, x_1, x_2)$ coordinated approximation for problem 3.2.2

Stationarity conditions successively for $v, x_1, x_2, \lambda_1, \lambda_1$ and $\lambda_3$ with the

feedback function $Q(\tau, s) = \tau \ln s$ will be

$$\left\{\begin{array}{rcl} -1 + \overline{\lambda}_1 + \overline{\lambda}_2 &=& 0\,, \\ \overline{\lambda}_2 - \overline{\lambda}_3 &=& 0\,, \\ -\overline{\lambda}_1 + \overline{\lambda}_3 &=& 0\,, \\ -\overline{x}_2 + \alpha + \overline{v} + \tau \ln \overline{\lambda}_1 &=& 0\,, \\ -\beta + \overline{x}_1 + \overline{v} + \tau \ln \overline{\lambda}_2 &=& 0\,, \\ -\overline{x}_1 + \overline{x}_2 + \tau \ln \overline{\lambda}_3 &=& 0\,. \end{array}\right. \tag{3.2.8}$$

From (3.2.8) we obtain $\overline{v} = \tau \ln \left( e^{\frac{\overline{x}_2 - \alpha}{\tau}} + e^{\frac{-\overline{x}_1 + \beta}{\tau}} \right)$ and therefore $\overline{x}_2 - \overline{x}_1 = \tau \ln 2 > 0$, which gives for the dependence $v_{x_1}$ the approximation

$$\overline{v}(\tau, x_1) = \tau \ln \left( 2e^{\frac{x_1 - \alpha}{\tau}} + e^{\frac{-x_1 + \beta}{\tau}} \right)\,,$$

the graph of which is shown with values for the parameters $\alpha = 1$, $\beta = 5$ and $\tau = 0.5$, 0.2, 0.01 in Fig. 3.2.1.

It is easy to verify that this approximation of the maximum function has a minimum at the point

$$x_1(\tau) = \frac{\alpha + \beta}{2} + \tau \ln \sqrt{2}\,.$$

Note that problem 3.2.2 in the usual sense has no solution. But the minimax under study has an *infimum*, equal to $\dfrac{\alpha + \beta}{2} = 3\,.$

In conclusion, we note that the use in the described scheme to take into account the constraint $x_1 \le x_2$ of any sufficiently smooth penalty function allows us to obtain an approximation of the two-dimensional dependence $v_{x_1, x_2}$ as well. The form of the isoline system for such an approximation for $\alpha = -5$, $\beta = 11$ and $\tau = 0.05$ is shown in Fig. 3.2.2.

A sufficiently high degree of smoothness of the approximation $\overline{v}(\tau, X)$ allows the use of the *standard* classical set of tools for searching and analyzing stationary points of this function.

We will demonstrate this for the following problem.

`Example 3.2.3.` Find in $E^2$ the minimum value of the function

$$Y(x_1, x_2) = \max\left\{ x_1^2 + x_2^2;\quad 25 - (x_1+1)^2 - (x_2-2)^2 \right\}.$$

`Solution.` The statement of the problem (3.2.10) in this case has the form.

*maximize with respect to* $\{x_1, x_2, v\}$ *function* $-v$, *subject to*

$$\begin{aligned} x_1^2 + x_2^2 - v &\leq 0, \\ 25 - (x_1+1)^2 - (x_2-2)^2 - v &\leq 0. \end{aligned}$$

Helper function (3.2.11) in this task there will be

$$U(\tau, x_1, x_2, \mu_1, \mu_2, v) = -v - \mu_1\left(x_1^2 + x_2^2 - v\right) -$$

$$-\mu_2\big(25 - (x_1+1)^2 - (x_2-2)^2 - v\big) + R(\tau, \mu_1) + R(\tau, \mu_2). \qquad (3.2.14)$$

Stationarity conditions for (3.2.14) with respect to $x_1, x_2, \mu_1, \mu_2$ and $v$ are:

$$\left\{\begin{array}{rcl} -2\overline{\mu}_1\overline{x}_1 + 2\overline{\mu}_2\left(\overline{x}_1+1\right) &=& 0, \\ -2\overline{\mu}_1\overline{x}_2 + 2\overline{\mu}_2\left(\overline{x}_2-2\right) &=& 0, \\ -\left(\overline{x}_1^2 + \overline{x}_2^2 - \overline{v}\right) + Q(\tau, \overline{\mu}_1) &=& 0, \\ -\left(25 - \left(\overline{x}_1+1\right)^2 - \left(\overline{x}_2-2\right)^2 - \overline{v}\right) + Q(\tau, \overline{\mu}_2) &=& 0, \\ -1 + \overline{\mu}_1 + \overline{\mu}_2 &=& 0, \end{array}\right. \qquad (3.2.15)$$

where $Q(\tau, s) = \tau \ln s$.

We briefly describe the solution scheme for system (3.2.15). From the first two equations we have $\overline{x}_2 = -2\overline{x}_1$. From the last three equations it follows that as a smoothed approximation of the function $Y\left(\overline{x}_1, \overline{x}_2\right)$ we can take

$$\overline{v}\left(\tau, x_1, x_2\right) = \tau \ln\left(\exp\left(\frac{x_1^2 + x_2^2}{\tau}\right) + \exp\left(\frac{25 - \left(x_1+1\right)^2 - \left(x_2-2\right)^2}{\tau}\right)\right). \qquad (3.2.16)$$

The system of isolines of (3.2.16) for $\tau = 0.1$ is shown in Fig. 3.2.4.

Based on the type of isolines of function (3.2.16), we can conclude that this function has stationary points $\left\{\overline{x}_{1(A)}; \overline{x}_{2(A)}\right\}$ , $\left\{\overline{x}_{1(B)}; \overline{x}_{2(B)}\right\}$ and $\left\{\overline{x}_{1(C)}; \overline{x}_{2(C)}\right\}$. These points belong respectively to the neighborhoods of points $A, B$ and $C$, shown in Fig. 3.2.4.

In turn, the coordinates of points $A, B$ and $C$ can be found using the definition of the function $Y(x_1, x_2)$. Point $B$ with coordinates $\{-1; 2\}$ is a local maximum of the function $25 - (x_1 + 1)^2 - (x_2 - 2)^2$ .

Points A and C are the solution to the problem of conditional extremum:

$$\begin{aligned} &\textit{maximize by } \{x_1, x_2\} \qquad x_1^2 + x_2^2 \\ &\textit{provided} \qquad\quad 25 - (x_1 + 1)^2 - (x_2 - 2)^2 = x_1^2 + x_2^2 . \end{aligned}$$

It is easy to verify that its solutions are $A\{-2\,;\,4\}$ and $C\{1\,;\,-2\}$.

T a b l e 3.2.2a

Neighborhood of $A$ with initial point: $\overline{x}_1 = -1.5$; $\overline{x}_2 = 3.5$; $\overline{v} = 22.5$

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{v}$ |
|---|---|---|---|
| 1.00 | 1.976305906 | 3.952611813 | 20.63559463 |
| 0.95 | -1.977519843 | 3.955039686 | 20.60385998 |
| 0.90 | -1.978730572 | 3.957461144 | 20.57212044 |
| 0.85 | -1.979938116 | 3.959876232 | 20.54037603 |
| 0.80 | -1.981142498 | 3.962284995 | 20.50862678 |
| *Solution* | -2 | 4 | 20 |

T a b l e 3.2.2b

Neighborhood of $B$ with initial point: $\overline{x}_1 = -0.6$; $\overline{x}_2 = 2.3$; $\overline{v} = 24.8$

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{v}$ |
|---|---|---|---|
| 1.00 | -1.000000002 | 2.000000004 | 25 |
| 0.95 | -1.000000001 | 2.000000001 | 25 |
| 0.90 | -1 | 2 | 25 |
| 0.85 | -1 | 2 | 25 |
| 0.80 | -1 | 2 | 25 |
| *Solution* | -1 | 2 | 25 |

T a b l e 3.2.2c

Neighborhood of $C$ with initial point: $\overline{x}_1 = 3.0$; $\overline{x}_2 = 1.0$; $\overline{v} = 10.0$

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{v}$ |
|---|---|---|---|
| 1.00 | 1.022565341 | -2.045130681 | 5.63737652 |
| 0.95 | 1.021461696 | -2.042923391 | 5.605467921 |
| 0.90 | 1.020355555 | -2.040711109 | 5.573563393 |
| 0.85 | 1.019246903 | -2.038493805 | 5.541662957 |
| 0.80 | 1.018135724 | -2.036271447 | 5.50976663 |
| *Solution* | 1 | -2 | 5 |

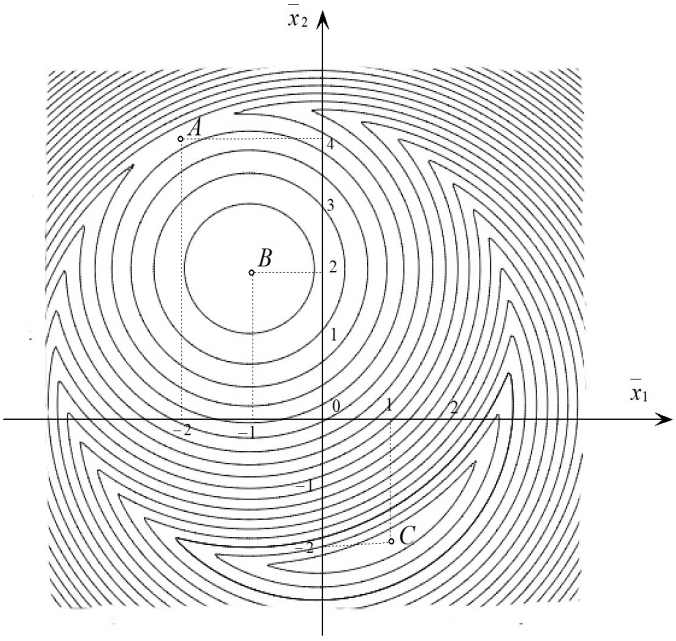



Fig. 3.2.4. The system of isolines of the function (3.2.16)

Tables 3.2.2a, 3.2.2b and 3.2.2c present numerical estimates of the coordinates of the stationary points $\{\overline{x}_{1(A)}; \overline{x}_{2(A)}\}$, $\{\overline{x}_{1(B)}; \overline{x}_{2(B)}\}$ and

$\{\overline{x}_{1(C)}; \overline{x}_{2(C)}\}$ for the approximating function $\overline{v}(\tau, x_1, x_2)$ for different values of the parameter $\tau$.

To illustrate the smoothing property of the feedback functions, we use the equality $\overline{x}_2 = -2\overline{x}_1$, which follows from system (3.2.15). In this case, the dependence of $\overline{v}$ on $\tau$ and $\overline{x}_1$ can be represented by the function

$$\Phi(\tau, \overline{x}_1) = \tau \ln \left( \exp \frac{5\overline{x}_1^2}{\tau} + \exp \left( \frac{25 - 5\left(\overline{x}_1 + 1\right)^2}{\tau} \right) \right). \qquad (3.2.17)$$

The graph of this function, as well as its fragments in the neighborhoods of stationary points for different values of the parameter $\tau$, are shown in Fig. 3.2.5, 3.2.6, 3.2.7 and 3.2.8.

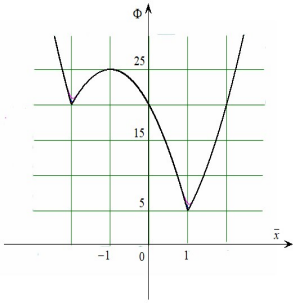



Fig. 3.2.5. Graphs of the function $\Phi(\tau, \overline{x})$

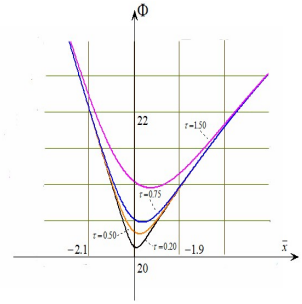



Fig. 3.2.6. Function graphs $\Phi(\tau, \overline{x})$ in the neighborhood of point $A$

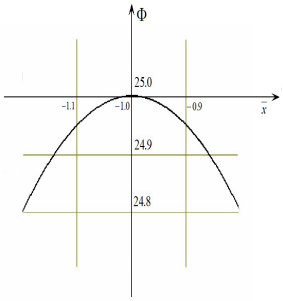



Fig. 3.2.7. Graphs of the function $\Phi(\tau, \overline{x})$ in the neighborhood of point $B$

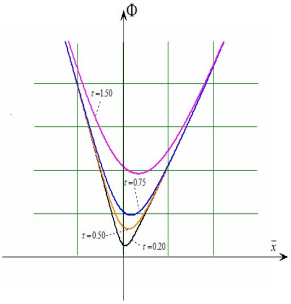



Fig. 3.2.8. Graphs of the function $\Phi(\tau, \overline{x})$ in the neighborhood of point $C$

Note that in the neighborhood of point $B$ (see Fig. 3.2.7) the differences in the values of the smoothing function for different $\tau$ are small and are

practically not noticeable on the graph. This effect is explained by the specific properties of the feedback function $Q(\tau, s) = \tau \ln s$, used in the formulation of system (3.2.15).

To study the behavior of the function $\overline{v}\,(\tau, x_1, x_2)$ we use the standard sufficient conditions for the classification of stationary points, based on the Sylvester criterion. For this purpose, we will first give formulas for the partial derivatives of this function up to and including the second order.

We introduce the notation

$$\Delta = \exp \frac{x_1^2 + x_2^2}{\tau} + \exp \frac{25 - (x_1 + 1)^2 - (x_2 - 2)^2}{\tau}.$$

Then

$$\frac{\partial \overline{v}}{\partial x_1} = -\frac{2}{\Delta}\left((1 + x_1) \exp \frac{25 - (x_1 + 1)^2 - (x_2 - 2)^2}{\tau} - x_1 \exp \frac{x_1^2 + x_2^2}{\tau}\right)$$

And

$$\frac{\partial \overline{v}}{\partial x_2} = 2x_2 + \frac{4(1 - x_2)}{\Delta} \exp \frac{25 - (x_1 + 1)^2 - (x_2 - 2)^2}{\tau}.$$

Similarly, for the second derivatives we have

$$\frac{\partial^2 \overline{v}}{\partial x_1^2} = \frac{2}{\Delta^2}\left(\frac{2(1 + 2x_1)^2}{\tau} \exp \frac{-2x_1 + 4x_2 + 20}{\tau} - \right.$$

$$\left. - \exp \frac{2}{\tau}\Big(25 - (x_1 + 1)^2 - (x_2 - 2)^2\Big) + exp \frac{2}{\tau}\Big(x_1^2 + x_2^2\Big)\right),$$

$$\frac{\partial^2 \overline{v}}{\partial x_1 \partial x_2} = \frac{8(2x_1 + 1)(x_2 - 1)}{\tau \Delta^2} \exp \frac{-2x_1 + 4x_2 + 20}{\tau},$$

$$\frac{\partial^2 \overline{v}}{\partial x_2^2} = \frac{2}{\Delta^2}\left(\frac{8(1 - x_2)^2}{\tau} \exp \frac{-2x_1 + 4x_2 + 20}{\tau} - \right.$$

$$\left. - \exp \frac{2}{\tau}\Big(25 - (x_1 + 1)^2 - (x_2 - 2)^2\Big) + \exp \frac{2}{\tau}\Big(x_1^2 + x_2^2\Big)\right).$$

At points $\left\{\overline{x}_{1(A)}; \overline{x}_{2(A)}\right\}$, $\left\{\overline{x}_{1(B)}; \overline{x}_{2(B)}\right\}$ and $\left\{\overline{x}_{1(C)}; \overline{x}_{2(C)}\right\}$ the Hessian matrices of the function $\overline{v}(\tau, \overline{x}_1, \overline{x}_2)$ for $\tau = 0.85$ will be

$$\|\mathrm{Hess}(A)\| = \left\| \begin{array}{cc} 8.454728203 & -18.260864263 \\ -18.260864263 & 35.846024597 \end{array} \right\|,$$

$$\|\mathrm{Hess}(B)\| = \left\| \begin{array}{cc} -1.999999999 & -5.688 \cdot 10^{-10} \\ -5.688 \cdot 10^{-10} & -1.999999999 \end{array} \right\|,$$

$$\|\mathrm{Hess}(C)\| = \left\| \begin{array}{cc} 10.343449802 & -19.370457876 \\ -19.370457876 & 39.399136601 \end{array} \right\|.$$

It is known from the course of mathematical analysis that to use the Sylvester criterion, as a sufficient condition for the presence or absence of an extremum, the following values are necessary:

$$\text{gradient norm} \quad Ngr = \sqrt{\left(\frac{\partial \overline{v}}{\partial x_1}\right)^2 + \left(\frac{\partial \overline{v}}{\partial x_2}\right)^2},$$

as well as the left upper principal minors of the Hessian matrix

$$M_{(1)} = \frac{\partial^2 \overline{v}}{\partial x_1^2} \qquad \text{and} \qquad M_{(2)} = \frac{\partial^2 \overline{v}}{\partial x_1^2}\frac{\partial^2 \overline{v}}{\partial x_2^2} - \left(\frac{\partial^2 \overline{v}}{\partial x_1 \partial x_2}\right)^2.$$

The values of these characteristics for stationary points $\left\{\overline{x}_{1(A)}; \overline{x}_{2(A)}\right\}$, $\left\{\overline{x}_{1(B)}; \overline{x}_{2(B)}\right\}$ and $\left\{\overline{x}_{1(C)}; \overline{x}_{2(C)}\right\}$, calculated for $\tau = 0.85$, as well as the classification of these points according to the Sylvester criterion are given in Table 3.2.3.

Note that the comparatively large value of the relative error of the obtained solutions is due to the properties of the function

$$Q(\tau, s) = \tau \ln s.$$

This kind of the function was used only to present solutions in a more readable form.

If we limit ourselves to a numerical search for stationary points, then it is enough to use the feedback function

$$Q(\tau, s) = \tau\left(s - \frac{1}{s}\right).$$

This allows obtaining solutions with a significantly smaller error only by decreasing the value of the parameter $\tau$ in system (3.2.15).

T a b l e 3.2.3

| | Solutions of system (3.2.15) in the neighborhood of the point: | | |
|---|---|---|---|
| | $A$ | $B$ | $C$ |
| $\overline{x}_1$ | -1.979938116 | -1.000000000 | 1.019246903 |
| $\overline{x}_2$ | 3.959876232 | 2.000000000 | -2.038493805 |
| $\overline{v}$ | 20.540376029 | 25.000000000 | 5.541662957 |
| $Ngr$ | $3.2095 \cdot 10^{-9}$ | $0.2418 \cdot 10^{-9}$ | $9.4269 \cdot 10^{-9}$ |
| $M_{(1)}$ | 8.454728203 | -1.999999999 | 10.343449802 |
| $M_{(2)}$ | -30.390768491 | 3.999999996 | 32.308353346 |
| *Point type*: | No extremum (saddle) | Local maximum | Local minimum |

The results of solving system (3.2.15) in small neighborhoods of points $\left\{\overline{x}_{1(A)}; \overline{x}_{2(A)}\right\}$ , $\left\{\overline{x}_{1(B)}; \overline{x}_{2(B)}\right\}$ and $\left\{\overline{x}_{1(C)}; \overline{x}_{2(C)}\right\}$ for this case are given in Tables 3.2.4($A$), 3.2.4($B$) and 3.2.4($C$).

T a b l e 3.2.4($A$)

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{\mu}_1$ | $\overline{\mu}_2$ | $\overline{v}$ |
|---|---|---|---|---|---|
| 1.00 | −1.934001409 | 3.868002818 | 0.325662654 | 0.674337346 | 21.44680679 |
| $10^{-1}$ | −1.993845902 | 3.987691805 | 0.332646728 | 0.667353272 | 20.14446196 |
| $10^{-2}$ | −1.999388464 | 3.998776928 | 0.333265357 | 0.666734643 | 20.01444462 |
| $10^{-3}$ | −1.999938885 | 3.999877769 | 0.333326542 | 0.666673458 | 20.00144445 |
| $10^{-4}$ | −1.999993889 | 3.999987778 | 0.333332654 | 0.666667346 | 20.00014444 |
| $10^{-5}$ | −1.999999389 | 3.999998778 | 0.333333265 | 0.666666735 | 20.00001444 |
| $10^{-6}$ | −1.999999939 | 3.999999878 | 0.333333327 | 0.666666673 | 20.00000144 |
| $10^{-7}$ | −1.999999994 | 3.999999988 | 0.333333333 | 0.666666667 | 20.00000014 |
| $10^{-8}$ | −1.999999999 | 3.999999999 | 0.333333333 | 0.66666667 | 20.00000001 |
| $10^{-9}$ | −2 | 4 | 0.333333333 | 0.666666667 | 20 |
| $10^{-10}$ | −2 | 4 | 0.333333333 | 0.666666667 | 20 |
| *Exact.* | −2 | 4 | $\frac{1}{3}$ | $\frac{2}{3}$ | 20 |

If it is not possible to select a feedback function that provides an acceptable error for a given $\tau$, then it is possible to apply the procedure of iterative refinement of the solution using formulas similar to (3.6):

$$\overline{x}_{j(T+1)} = \overline{x}_{j(T)} - \tau \frac{d\overline{x}_{j(T)}}{d\tau} \qquad \forall j = 1, 2 \qquad \forall T = 0,\ 1,\ 2,\ \ldots \quad (3.2.18)$$

where $T$ is the iteration number.

Since $\tau$ is now fixed, then for the refining vector we can introduce the notation $\delta\overline{x} = \tau\dfrac{d\overline{x}}{d\tau}$.

Then, according to the implicit function theorem, the components of this vector satisfy a system of linear equations of the form:

$$\begin{cases} \dfrac{\partial^2\overline{v}}{\partial x_1^2}\,\delta\overline{x}_{1(T)} + \dfrac{\partial^2\overline{v}}{\partial x_1\partial x_2}\,\delta\overline{x}_{2(T)} = -\tau\dfrac{\partial^2\overline{v}}{\partial x_1\partial\tau}, \\ \dfrac{\partial^2\overline{v}}{\partial x_1\partial x_2}\,\delta\overline{x}_{1(T)} + \dfrac{\partial^2\overline{v}}{\partial x_1^2}\,\delta\overline{x}_{2(T)} = -\tau\dfrac{\partial^2\overline{v}}{\partial x_2\partial\tau}, \end{cases}$$

Where

$$\frac{\partial^2\overline{v}}{\partial x_1\partial\tau} = -\frac{4}{\tau^2\Delta^2}(1+2x_1)\left(x_1^2+x_1+x_2^2-2x_2-10\right)\exp\frac{-2x_1+4x_2+20}{\tau},$$

$$\frac{\partial^2\overline{v}}{\partial x_2\partial\tau} = -\frac{8}{\tau^2\Delta^2}(x_2-1)\left(x_1^2+x_1+x_2^2-2x_2-10\right)\exp\frac{-2x_1+4x_2+20}{\tau}.$$

T a b l e 3.2.4($B$)

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{\mu}_1$ | $\overline{\mu}_2$ | $\overline{v}$ |
|---|---|---|---|---|---|
| 1.00 | −1.056966891 | 2.113933783 | 0.051140285 | 0.948859715 | 25.08881072 |
| $10^{-1}$ | −1.005063133 | 2.010126265 | $5.0124\cdot10^{-3}$ | 0.994987624 | 25.00087682 |
| $10^{-2}$ | −1.000500626 | 2.001001251 | $5.0012\cdot10^{-4}$ | 0.999499875 | 25.00000875 |
| $10^{-3}$ | −1.000050006 | 2.000100013 | $5.0001\cdot10^{-5}$ | 0.999949999 | 25.00000009 |
| $10^{-4}$ | −1.000005000 | 2.000010000 | $5.0000\cdot10^{-6}$ | 0.999995000 | 25 |
| $10^{-5}$ | −1.000000500 | 2.000001000 | $5.0000\cdot10^{-7}$ | 0.999999500 | 25 |
| $10^{-6}$ | −1.000000050 | 2.000000100 | $5.0000\cdot10^{-8}$ | 0.999999950 | 25 |
| $10^{-7}$ | −1.000000005 | 2.000000010 | $5.0000\cdot10^{-9}$ | 0.999999995 | 25 |
| $10^{-8}$ | −1.000000001 | 2.000000001 | $5.0000\cdot10^{-10}$ | 0.999999999 | 25 |
| $10^{-9}$ | −1 | 2 | $5.0000\cdot10^{-11}$ | 1 | 25 |
| $10^{-10}$ | −1 | 2 | $5.0000\cdot10^{-12}$ | 1 | 25 |
| *Точн. реш.* | −1 | 2 | 0 | 1 | 25 |

Based on the assumption of continuity of the second derivatives of the function $\overline{v}(\tau, x_1 < x_2)$ with respect to all its arguments in some compact set containing the point $C$, we can assume that the process (3.2.18) will converge to the point $C$ for a sufficiently small positive $\tau$ by virtue of the principle of contraction transformations.

T a b l e 3.2.4($C$)

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{\mu}_1$ | $\overline{\mu}_2$ | $\overline{v}$ |
|---|---|---|---|---|---|
| 1.00 | 1.057334324 | −2.114668648 | 0.660530720 | 0.339469280 | 6.443182771 |
| $10^{-1}$ | 1.006069204 | −2.012138408 | 0.665995028 | 0.334004972 | 5.144427985 |
| $10^{-2}$ | 1.000610687 | −2.001221375 | 0.666598840 | 0.333401160 | 5.014444275 |
| $10^{-3}$ | 1.000061107 | −2.000122214 | 0.666659877 | 0.333340123 | 5.001444443 |
| $10^{-4}$ | 1.000006111 | −2.000012222 | 0.666665988 | 0.333334012 | 5.000144444 |
| $10^{-5}$ | 1.000000611 | −2.000001222 | 0.666666599 | 0.333333401 | 5.000014444 |
| $10^{-6}$ | 1.000000061 | −2.000000122 | 0.666666660 | 0.333333340 | 5.000001444 |
| $10^{-7}$ | 1.000000006 | −2.000000012 | 0.666666666 | 0.333333334 | 5.000000144 |
| $10^{-8}$ | 1.000000001 | −2.000000001 | 0.666666667 | 0.333333333 | 5.000000014 |
| $10^{-9}$ | 1 | −2 | 0.666666667 | 0.333333333 | 5.000000001 |
| $10^{-10}$ | 1 | −2 | 0.666666667 | 0.333333333 | 5 |
| *Exact. sol.* | 1 | −2 | $\frac{2}{3}$ | $\frac{1}{3}$ | 5 |

The results of the corresponding calculations for the problem being solved are given in Table. 3.2.5a and 3.2.5b, in which the notation is used: $\overline{F} = \overline{x}_1^2 + \overline{x}_2^2$ and $\overline{G} = 25 - (\overline{x}_1 - 1)^2 - (\overline{x}_2 + 2)^2$ .

T a b l e 3.2.5a

| $T$ | $\overline{x}_{1(T)}$ | $\overline{x}_{2(T)}$ | $\delta\overline{x}_{1(T)}$ | $\delta\overline{x}_{2(T)}$ |
|---|---|---|---|---|
| 0 | 1.019246903 | −2.038493805 | −0.018868520 | 0.037737040 |
| 1 | 1.000378383 | −2.000756765 | $-3.7824 \cdot 10^{-4}$ | $7.5648 \cdot 10^{-4}$ |
| 2 | 1.000000143 | −2.000000286 | $-1.4321 \cdot 10^{-7}$ | $2.8612 \cdot 10^{-7}$ |
| 3 | 1 | −2 | ... | ... |
| $\infty$ | 1 | −2 | 0 | 0 |

T a b l e 3.2.5b

| $T$ | $\overline{x}_{1(T+1)}$ | $\overline{x}_{2(T+1)}$ | $\overline{F}_{(T+1)}$ | $\overline{G}_{(T+1)}$ |
|---|---|---|---|---|
| 0 | 1.000378383 | −2.000756765 | 5.003784542 | 4.992431632 |
| 1 | 1.000000143 | −2.000000286 | 5.000001431 | 4.999997138 |
| 2 | 1 | −2 | 5 | 5 |
| 3 | ... | ... | ... | ... |
| $\infty$ | 1 | −2 | 5 | 5 |

## 3.3. Finding minimax and maximin on a discrete set

Formulas (3.1.4) also allow solving more complex optimization problems.

For example, the method of feedback functions is applicable to solving problems whose objective functions are superpositions of operators for finding an extremum on a discrete set.

Such problems include the problem of finding minimax and/or maximin on the set of values of elements of the matrix

$$\|A_{ij}\| = \left\| \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right\|,$$

where the maximum element is selected from each row, and the minimum from each column. In other words, we need to find

$$A_{\text{maxmin}} = \max_{j=\overline{1,n}} \Big( \min_{i=\overline{1,m}} A_{ij} \Big) \qquad \text{and} \qquad A_{\text{minmax}} = \min_{i=\overline{1,m}} \Big( \max_{j=\overline{1,n}} A_{ij} \Big). \tag{3.3.1}$$

To solve problems (3.3.1) we use equalities (3.1.4), according to which

$$A_{\text{maxmin}} = \lim_{\tau \to +0} \tau \ln \Big( \sum_{i=1}^{m} \exp \left( \tfrac{A_{i,\min}}{\tau} \right) \Big),$$

where

$$A_{i,\min} = - \lim_{\tau \to +0} \tau \ln \Big( \sum_{j=1}^{n} e^{-\frac{A_{ij}}{\tau}} \Big).$$

Since the expressions under the limit sign are continuous functions, we can first find their superposition, then pass to the limit. Therefore, we have

$$A_{\text{maxmin}} = \lim_{\tau \to +0} \tau \ln \left( \sum_{i=1}^{m} \exp \left( \frac{1}{\tau} \cdot \tau \ln \left( \left( \sum_{j=1}^{n} e^{-\frac{A_{ij}}{\tau}} \right)^{-1} \right) \right) \right).$$

Where, after obvious simplifications, we get

$$A_{\text{maxmin}} = \lim_{\tau \to +0} \tau \ln \sum_{i=1}^{m} \Big( \sum_{j=1}^{n} e^{-\frac{A_{ij}}{\tau}} \Big)^{-1} \tag{3.3.2}$$

and, similarly,

$$A_{\text{minmax}} = -\lim_{\tau\to+0} \tau \ln \sum_{j=1}^{n}\Big(\sum_{i=1}^{m} e^{\frac{A_{ij}}{\tau}}\Big)^{-1}. \qquad (3.3.3)$$

Let, for example, the matrix $\|A\|$ have the form

$$\|A_{ij}\| = \begin{Vmatrix} 5 & 2 & 8 & 4 \\ 12 & 3 & 11 & 9 \\ 1 & 6 & 7 & 10 \end{Vmatrix}.$$

It is easy to verify that in this case $A_{\text{maxmin}} = 7$ and $A_{\text{minmax}} = 8$. As an illustration of the use of formulas (3.3.2) and (3.3.3), for this matrix, in Tables 3.3.1a and 3.3.1b the values of the functions standing in (3.3.2) and (3.3.3) under the limit sign are given, for different values of the parameter $\tau$.

T a b l e 3.3.1a.

| $\tau$ | 0.300 | 0.250 | 0.200 | 0.150 |
|---|---|---|---|---|
| $A_{\text{minmax}}$ | 7.999631605 | 7.999917657 | 7.999990981 | 7.999999757 |
| $A_{\text{maxmin}}$ | 6.989497444 | 6.995464027 | 6.998656991 | 6.999809227 |

T a b l e 3.3.1b.

| $\tau$ | 0.125 | 0.100 | 0.075 | 0.050 |
|---|---|---|---|---|
| $A_{\text{minmax}}$ | 7.999999986 | 8 | 8 | 8 |
| $A_{\text{maxmin}}$ | 6.999958074 | 6.99999546 | 6.999999879 | 7 |

Let us now consider an example of applying the method of feedback functions to a more general setting of the problem of finding superpositions of extremum operators.

Let it be required to find the maximin and minimax for a matrix function depending on two scalar parameters $v_1$ and $v_2$

$$\|D_{ij}(v_1, v_2)\| = \begin{Vmatrix} 1 & 25 - v_1 - v_2 & 3 & 4 \\ 5 & v_1 & 7 & 8 \\ 9 & v_2 & 11 & 12 \end{Vmatrix}, \qquad (3.3.4)$$

if $0 \le v_1 \le 15$ and $0 \le v_2 \le 15$.

It can be shown that in this problem the dependencies

$$D_{\mathrm{minmax}_{v_1 v_2}} = \min_{i=\overline{1,3}} \Big( \max_{j=\overline{1,4}} D_{ij}(v_1, v_2) \Big)$$

and

$$D_{\mathrm{maxmin}_{v_1 v_2}} = \max_{j=\overline{1,4}} \Big( \min_{i=\overline{1,3}} D_{ij}(v_1, v_2) \Big)$$

are continuous, piecewise linear functions of $v_1$ and $v_2$.

To solve this problem, we will use the method of feedback functions. In other words, we will find, based on (3.3.2) and (3.3.3), smooth approximations for the dependencies $D_{\mathrm{minmax}_{v_1 v_2}}$ and $D_{\mathrm{maxmin}_{v_1 v_2}}$.

$$\overline{D}_{\mathrm{minmax}}(\tau, v_1, v_2) = -\tau \ln \sum_{j=1}^{4} \left( \sum_{i=1}^{3} e^{\frac{D_{ij}(v_1, v_2)}{\tau}} \right)^{-1} \tag{3.3.5}$$

$$\overline{D}_{\mathrm{maxmin}}(\tau, v_1, v_2) = \tau \ln \sum_{i=1}^{3} \left( \sum_{j=1}^{4} e^{-\frac{D_{ij}(v_1, v_2)}{\tau}} \right)^{-1}. \tag{3.3.6}$$

For the functions $\overline{D}_{\mathrm{minmax}}(\tau, v_1, v_2)$ and $\overline{D}_{\mathrm{maxmin}}(\tau, v_1, v_2)$ the systems of isolines for the value $\tau = 0.05$ are shown in Fig. 3.3.1 and 3.3.2, respectively.

These figures also indicate the values of these functions at characteristic points.

Note that the problem of finding, for example, $A_{\mathrm{minmax}}$ can be formulated as a linear programming problem

$$\begin{aligned} &\textit{maximize with respect to } \{Y, f\} \quad \textit{function} \quad f \\ &\textit{subject to} \quad \alpha_{ij} - y_i \le 0 \qquad j = \overline{1, n}, \, , \\ &\qquad\qquad y_i + f \le 0 \qquad\quad i = \overline{1, m}\,. \end{aligned} \tag{3.3.7}$$

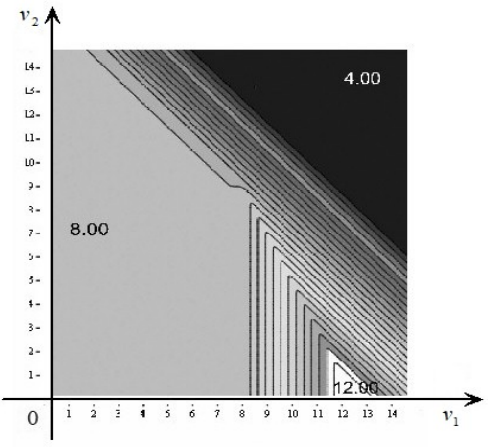



Fig. 3.3.1. System of isolines of function $\overline{D}_{\text{minmax}}(\tau, v_1, v_2)$

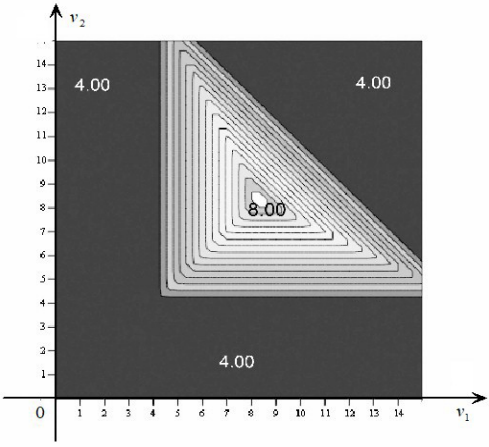



Fig. 3.3.2. System of isolines of the function $\overline{D}_{\text{maxmin}}(\tau, v_1, v_2)$

## 3.4. Problems for multicriteria models

From the formulation of the method of feedback functions it follows that it is possible to obtain in *explicit* form a smoothing approximation of the dependence $x_v^*$ only in exceptional cases.

In practice, however, it often turns out that when solving problems it is sufficient to know only its *Taylor polynomials*. For this, only the values of the approximating function $\overline{X}(\tau, V)\ \forall V \in \Upsilon$ and its partial derivatives up to a certain order are required.

We will show that this version of the method of feedback functions is applicable in solving optimization problems of the type (2.3.1) for parametric *multicriterial* mathematical models.

As is known, in mathematical modeling when forming criteria for assessing the states of the modeled object, not one, but a certain set of $T \geq 2$ independent objective functions often arises.

In this case, *multicriteria model* we will call the set of functions to be maximized with respect to $X \in E^n$

$$F_k(X, V) \qquad k = \overline{1, T}\,, \tag{3.4.1}$$

under the conditions

$$f_i(X,V) \le 0 \quad i = \overline{1,m}\,, \tag{3.4.2}$$

where $V \in \Upsilon \subseteq E^K$ is the vector of exogenous parameters.

We will also assume that all functions $F_k(X,V)$ and $f_i(X,V)$ are twice continuously differentiable with respect to all their arguments.

When making management decisions using multicriteria models, various types of optimization problems may arise.

For example, it is obvious that simultaneous achievement by all objective functions (3.4.1) of extrema at some point of the set, defined by the system of inequalities (3.4.2), is generally impossible. Therefore, in practice, problem statements are often used, the admissible solutions of which are some *compromise* for the criteria (3.4.1).

The basis for constructing a compromise can be solutions to single-criteria problems of the form

$$\begin{aligned} &\textit{maximize by } X \in E^n \qquad \textit{function } F_k(X,V) \\ &\textit{subject to:} \qquad\qquad\qquad f_i(X,V) \le 0 \quad i = \overline{1,m}. \end{aligned} \tag{3.4.3}$$

The solution to each of these problems (i.e. the vector $X^*_{(k)v}$, as well as the value $F^*_{(k)v} = F_k\big(X^*_{(k)v}, V\big)$ $k = \overline{1,T}$ is the best admissible state of the model (3.4.1)—(3.4.2) according to the $k$-th criterion for a fixed vector of parameters $V \in \Upsilon$.

We consider the case when the problem of determining a compromise state consists of finding an admissible point $X^{**}_v$, at which the extremum of some *quantitative assessment of the quality of the compromise* is achieved.

Such a point, for example, can be the vector $X$, satisfying constraints (3.4.2) and delivering *minimum* of the maximal (over $k = \overline{1,T}$) of the differences $F^*_{(k)v} - F_k(X,V)$. That is, $X$ is a solution to the problem

$$\begin{aligned} &\textit{maximize over } \{X \in E^n; \quad \rho \ge 0\} \ \textit{function} \quad -\rho \\ &\textit{under the conditions:} \qquad f_i(X,V) \le 0 \qquad\qquad i = \overline{1,m}, \\ &\qquad\qquad F^*_{(k)v} - F_k(X,V) - \rho \le 0 \qquad k = \overline{1,T}. \end{aligned} \tag{3.4.4}$$

More precisely, the solution to problem (3.4.4) is formed by the quantity $\rho^{**}_v$ and the vector $X^{**}_v$. They are some dependencies on the parameter vector $V$. These dependencies have the properties noted in § 2.3 that complicate the solution of problems in which they are used.

We also note that the value of $\rho_v^{**}$ can be considered as an estimate minimum of possible *discrepancy* of objective functions in the multicriterial model (3.4.1)—(3.4.2).

Further, for brevity, problems (3.4.3) will be called problems of the *first level*, and problem (3.4.4) a problem of the *second level*.

Other examples of second-level problems, in the formulation of which solutions to problems (3.4.3) are used, can also serve as well-known (see, for example, [15]) in the practice of mathematical modeling: algorithms *convolution of objective functions* (3.4.1), methods *search for Pareto equilibrium* and other computational schemes.

For the sake of completeness, we note that for the model (3.4.1)—(3.4.2) it is possible to formulate optimization problems of the *third level*, for example

$$\textit{find extremum } \rho_v^{**} \quad \textit{by } V \in \Upsilon \subseteq E^K . \qquad ()$$

The solution to this problem will be the vector $V^{***} \in \Upsilon$ and the number $\rho^{***}$, equal to th3.4.5e value of $\rho_v^{**}$ at the point $V^{***}$.

## 3.4.1. Method of feedback functions for solving optimization problems for multicriteria models

One of the reasons for the computational difficulties in solving problem (3.4.5) is the dependence $\rho_v^{**}$ in the formulation of this problem.

This problem can be solved by applying the method of feedback functions. That is, by constructing the function $\overline{\rho}(\tau, V)$, which is a smooth approximation of the dependence $\rho_v^{**}$.

Indeed, let a second-order method be used when searching for a local solution to problem (3.4.5). Therefore, for its implementation it is sufficient to have the values of $\overline{\rho}(\tau, V)$ and all its partial derivatives up to and including the second order.

According to § 2.3, the values of $\overline{\rho}(\tau, v)$ are found from the stationarity conditions for the auxiliary function (2.3.2). The values of the derivatives can be calculated using the implicit function theorem applied to these stationarity conditions.

Let us consider this approach specifically for the case of the multicriterial model (3.4.1)—(3.4.2).

First, we note that the condition of problem (3.4.4) contains nonsmooth dependencies $F_{(k)v}^* \quad k = \overline{1,T}$, which are solutions of problems

(3.4.3). Therefore, we will use the smoothing property of the method of feedback functions not only when solving the second-level problem (3.4.4), but also for problems (3.4.3).

Let the auxiliary function for the $k$-th problem (3.4.3)

$$U_{(k)}(\tau, X, \Lambda, V) = F_k(X, V) - \sum_{i=1}^{m} \lambda_i f_i(X, V) + \sum_{i=1}^{m} R(\tau, \lambda_i)\,. \qquad (3.4.6)$$

Then the stationarity conditions of function (3.4.6) are of the form

$$\left\{ \begin{array}{rcll} \dfrac{\partial U_{(k)}}{\partial x_p} & = & 0 & p = \overline{1,n}\,, \\ \dfrac{\partial U_{(k)}}{\partial \lambda_i} & = & 0 & i = \overline{1,m} \end{array} \right. \quad \text{or}$$

$$\left\{ \begin{array}{l} \operatorname*{grad}_x F_k(\overline{X}_{(k)}, V) - \sum\limits_{i=1}^{m} \overline{\lambda}_{(k)i} \operatorname*{grad}_x f_i(\overline{X}_{(k)}, V) \;=\; o\,, \\ \\ -f_i(\overline{X}_{(k)}, V) + Q(\tau, \overline{\lambda}_{(k)i}) = 0 \qquad i = \overline{1,m}\,. \end{array} \right. \qquad (3.4.7)$$

The solution of system (3.4.7) allows us to construct

$$\overline{U}_{(k)}(\tau, V) = U_{(k)}\Big(\tau, \overline{X}_{(k)}(\tau, V), \overline{\Lambda}_{(k)}(\tau, V), V\Big)\,. \qquad (3.4.8)$$

This smooth function can be used as an approximation of the $F^*_{(k)v}$ dependencies in the condition of the second-level problem (3.4.4).

By Theorem 2.1.3 the implicit function theorem can be applied to the stationarity conditions of the auxiliary function (3.4.7). This gives systems of linear equations $\forall k = \overline{1,T}$ for the values of the derivatives of the components of the vector functions $\overline{X}_{(k)}(\tau, V)$ and $\overline{\Lambda}_{(k)}(\tau, V)$ with respect to the parameters $v_t \quad t = \overline{1,K}$

$$\left\{ \begin{array}{ll} \sum\limits_{p=1}^{n} \dfrac{\partial^2 U_{(k)}}{\partial x_j \partial x_p} \dfrac{\partial \overline{x}_p}{\partial v_t} + \sum\limits_{q=1}^{m} \dfrac{\partial^2 U_{(k)}}{\partial x_j \partial \lambda_q} \dfrac{\partial \overline{\lambda}_q}{\partial v_t} = -\dfrac{\partial^2 U_{(k)}}{\partial x_j \partial v_t} & j = \overline{1,n}\,, \\ \\ \sum\limits_{p=1}^{n} \dfrac{\partial^2 U_{(k)}}{\partial \lambda_i \partial x_p} \dfrac{\partial \overline{x}_p}{\partial v_t} + \sum\limits_{q=1}^{m} \dfrac{\partial^2 U_{(k)}}{\partial \lambda_i \partial \lambda_q} \dfrac{\partial \overline{\lambda}_q}{\partial v_t} = -\dfrac{\partial^2 U_{(k)}}{\partial \lambda_i \partial v_t} & i = \overline{1,m}\,. \end{array} \right. \qquad (3.4.9)$$

The main matrix of the system of linear equations (3.4.9) is the Hessian matrix for the auxiliary function (3.4.6). Its elements, as well as the components of the column of the right-hand sides, are calculated at the point $\big\{\, \overline{X}_{(k)}(\tau, V),\, \overline{\Lambda}_{(k)}(\tau, V) \,\big\}$.

We now apply the method of feedback functions to solve the problem *of the second level* (3.4.4).

For ease of notation, we introduce the notation

$$\phi_k(\rho, X, V) = \overline{U}_{(k)}(\tau, V) - F_k(X, V) - \rho \qquad k = \overline{1, T}\,.$$

Then the condition of the second-level problem will take the form

$$\begin{array}{lrl} \textit{maximize over } \{X \in E^n, \rho \geq 0\} & -\rho & \\ \textit{under conditions:} & f_i(X, V) \leq 0 & i = \overline{1, m}, \\ & \phi_k(\rho, X, V) \leq 0 & k = \overline{1, T}. \end{array}$$

The helper function (2.3.2) for this task looks like this:

$$\mathbb{U}(\tau, \rho, X, \Lambda, \mu, V) =$$

$$= -\rho - \sum_{i=1}^{m} \lambda_i f_i(X, V) - \sum_{k=1}^{T} \mu_k \phi_k(\rho, X, V) - \tag{3.4.10}$$

$$-R(\tau, \rho) + \sum_{i=1}^{m} R(\tau, \lambda_i) + \sum_{k=1}^{T} R(\tau, \mu_k)\,,$$

where $\|\mu\| = \|\,\mu_1,\, \mu_2,\, \ldots,\, \mu_T\,\|$.

The stationarity conditions for the function $\mathbb{U}$ are of the form

$$\begin{cases} -1 + \sum\limits_{k=1}^{T} \overline{\overline{\mu}}_k - Q(\tau, \overline{\overline{\rho}}) = 0, \\ -\sum\limits_{i=1}^{m} \overline{\overline{\lambda}}_i \dfrac{\partial f_i}{\partial x_j}(\overline{\overline{X}}, V) - \sum\limits_{k=1}^{T} \overline{\overline{\mu}}_k \dfrac{\partial \phi_k}{\partial x_j}(\overline{\overline{\rho}}, \overline{\overline{X}}, V) = 0 \qquad j = \overline{1, n}\,, \\ -f_i(\overline{\overline{X}}, V) + Q(\tau, \overline{\overline{\lambda}}_i) = 0 \qquad i = \overline{1, m}\,, \\ -\phi_k(\overline{\overline{\rho}}, \overline{\overline{X}}, V) + Q(\tau, \overline{\overline{\mu}}_k) = 0 \qquad k = \overline{1, T}\,, \end{cases} \tag{3.4.11}$$

here $\{\ \overline{\overline{\rho}},\ \overline{\overline{X}},\ \overline{\overline{\Lambda}},\ \overline{\overline{\mu}}\ \}$ is a stationary point of the auxiliary function (3.4.10).

By Theorem 2.1.3, we can use the implicit function theorem for stationarity conditions of the function (3.4.10). This yields a system of linear equations (3.4.12), analogous to system (2.3.7). The solutions of system (3.4.12) are derivatives of $\{\ \overline{\overline{\rho}},\ \overline{\overline{X}},\ \overline{\overline{\Lambda}},\ \overline{\overline{\mu}}\ \}$ by parameters $v_t \quad t = \overline{1,k}$.

Recall that the implicit function theorem is also applicable to calculating derivatives of $\{\ \overline{\overline{\rho}},\ \overline{\overline{X}},\ \overline{\overline{\Lambda}},\ \overline{\overline{\mu}}\ \}$ by instrumental parameter $\tau$.

The values of these derivatives can be used in extrapolation procedures to reduce the error of the smoothed approximation by the formulas

$$\tilde{x}_j = \overline{x}_j(\tau, V) - \tau \frac{\partial \overline{x}_j}{\partial \tau} \quad j = \overline{1,n} \qquad \text{and} \qquad \tilde{\lambda}_i = \overline{\lambda}_i(\tau, V) - \tau \frac{\partial \overline{\lambda}_i}{\partial \tau} \quad i = \overline{1,m}\,.$$

$$\begin{cases} \dfrac{\partial^2 U}{\partial \rho^2}\dfrac{\partial \overline{\overline{\rho}}}{\partial v_t} + \displaystyle\sum_{j=1}^{n} \frac{\partial^2 U}{\partial \rho\, \partial x_j}\frac{\partial \overline{\overline{x}}_j}{\partial v_t} + \sum_{q=1}^{m} \frac{\partial^2 U}{\partial \rho\, \partial \lambda_q}\frac{\partial \overline{\overline{\lambda}}_q}{\partial v_t} + \sum_{k=1}^{T} \frac{\partial^2 U}{\partial \rho\, \partial \mu_k}\frac{\partial \overline{\overline{\mu}}_k}{\partial v_t} = -\dfrac{\partial^2 U}{\partial \rho\, \partial v_t}, \\ \dfrac{\partial^2 U}{\partial x_p\, \partial \rho}\dfrac{\partial \overline{\overline{\rho}}}{\partial v_t} + \displaystyle\sum_{j=1}^{n} \frac{\partial^2 U}{\partial x_p \partial x_j}\frac{\partial \overline{\overline{x}}_j}{\partial v_t} + \sum_{q=1}^{m} \frac{\partial^2 U}{\partial x_p \partial \lambda_q}\frac{\partial \overline{\overline{\lambda}}_q}{\partial v_t} + \sum_{k=1}^{T} \frac{\partial^2 U}{\partial x_p \partial \mu_k}\frac{\partial \overline{\overline{\mu}}_k}{\partial v_t} = -\dfrac{\partial^2 U}{\partial x_p \partial v_t} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p = \overline{1,n}\,, \\ \dfrac{\partial^2 U}{\partial \lambda_i\, \partial \rho}\dfrac{\partial \overline{\overline{\rho}}}{\partial v_t} + \displaystyle\sum_{j=1}^{n} \frac{\partial^2 U}{\partial \lambda_i \partial x_j}\frac{\partial \overline{\overline{x}}_j}{\partial v_t} + \sum_{q=1}^{m} \frac{\partial^2 U}{\partial \lambda_i \partial \lambda_q}\frac{\partial \overline{\overline{\lambda}}_q}{\partial v_t} + \sum_{k=1}^{T} \frac{\partial^2 U}{\partial \lambda_i \partial \mu_k}\frac{\partial \overline{\overline{\mu}}_k}{\partial v_t} = -\dfrac{\partial^2 U}{\partial \lambda_i \partial v_t} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = \overline{1,m}\,, \\ \dfrac{\partial^2 U}{\partial \mu_s\, \partial \rho}\dfrac{\partial \overline{\overline{\rho}}}{\partial v_t} + \displaystyle\sum_{j=1}^{n} \frac{\partial^2 U}{\partial \mu_s \partial x_j}\frac{\partial \overline{\overline{x}}_j}{\partial v_t} + \sum_{q=1}^{m} \frac{\partial^2 U}{\partial \mu_s \partial \lambda_q}\frac{\partial \overline{\overline{\lambda}}_q}{\partial v_t} + \sum_{k=1}^{T} \frac{\partial^2 U}{\partial \mu_s \partial \mu_k}\frac{\partial \overline{\overline{\mu}}_k}{\partial v_t} = -\dfrac{\partial^2 U}{\partial \mu_s \partial v_t} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s = \overline{1,T}\,, \end{cases} \tag{3.4.12}$$

Let us finally consider the problem *of the third level*. As a smooth approximation of the dependence $\rho_v^{**}$ we choose the function

$$\overline{\overline{\mathbb{U}}}(\tau, V) = \mathbb{U}\Big(\tau,\, \overline{\overline{\rho}}(\tau, V),\, \overline{\overline{X}}(\tau, V),\, \overline{\overline{\Lambda}}(\tau, V),\, \overline{\overline{\mu}}(\tau, V),\, V\Big)\,. \tag{3.4.13}$$

According to the rule of differentiation of a composite function, the partial

derivative of $\overline{\overline{\mathbb{U}}}$ with respect to $v_t \quad t = \overline{1,K}$ looks like

$$\frac{\partial\overline{\overline{\mathbb{U}}}}{\partial v_t} = \frac{\partial\mathbb{U}}{\partial v_t} + \frac{\partial\mathbb{U}}{\partial\rho}\frac{\partial\overline{\rho}}{\partial v_t} + \sum_{j=1}^{n}\frac{\partial\mathbb{U}}{\partial x_j}\frac{\partial\overline{x}_j}{\partial v_t} + \sum_{i=1}^{m}\frac{\partial\mathbb{U}}{\partial\lambda_i}\frac{\partial\overline{\lambda}_i}{\partial v_t} + \sum_{k=1}^{T}\frac{\partial\mathbb{U}}{\partial\mu_k}\frac{\partial\overline{\mu}_k}{\partial v_t}\,.$$

All partial derivatives of the function $\overline{\overline{\mathbb{U}}}$ in this formula are calculated at the point $\{\ \tau,\ \overline{\rho},\ \overline{\overline{X}},\ \overline{\overline{\Lambda}},\ \overline{\mu},\ V\ \}$. Therefore, by (3.4.11), this expression is simplified to

$$\frac{\partial\overline{\overline{\mathbb{U}}}}{\partial v_t} = \frac{\partial\mathbb{U}}{\partial v_t}\Big(\tau, \overline{\rho}(\tau,V),\ \overline{\overline{X}}(\tau,V),\ \overline{\overline{\Lambda}}(\tau,V),\ \overline{\mu}(\tau,V),\ V\Big) \qquad \forall t = \overline{1,K}\ . \tag{3.4.14}$$

Thus, to solve a third-level problem by any *first* order method, it is sufficient to be able to find only solutions to systems (3.4.7) and (3.4.11) $\forall V \in \Upsilon$. The approximation error depends on the value of the parameter $\tau$ and tends to zero at $\tau \to 0$.

In the case of using *second* order methods, in addition to solutions to systems (3.4.7) and (3.4.11), the values of the partial derivatives of the second-order approximating function (3.4.13) will also be required.

Direct differentiation of formula (3.4.14) taking into account (3.4.11) yields

$$\frac{\partial^2\overline{\overline{\mathbb{U}}}}{\partial v_s\partial v_t} = \frac{\partial^2\mathbb{U}}{\partial v_s\partial v_t} + \frac{\partial^2\mathbb{U}}{\partial v_s\partial\rho}\frac{\partial\overline{\rho}}{\partial v_t} + \sum_{j=1}^{n}\frac{\partial^2\mathbb{U}}{\partial v_s\partial x_j}\frac{\partial\overline{x}_j}{\partial v_t} + \sum_{i=1}^{m}\frac{\partial^2\mathbb{U}}{\partial v_s\partial\lambda_i}\frac{\partial\overline{\lambda}_i}{\partial v_t} +$$

$$+ \sum_{k=1}^{T}\frac{\partial^2\mathbb{U}}{\partial v_s\partial\mu_k}\frac{\partial\overline{\mu}_k}{\partial v_t} \qquad \forall s,t = \overline{1,K}\ . \tag{3.4.15}$$

It follows that in the case of using second-order methods to calculate the Hessian matrix $\nabla^2\overline{\overline{\mathbb{U}}}$ at each trial point $V \in \Upsilon$ we will also have to solve the linear system (3.4.12).

### 3.4.2. An example of a three-level problem for a multicriterial model

We will consider the use of the feedback function method to solve a three-level optimization parametric problem using a specific example.

On the one hand, this example is simple and easy to verify (here we have $K = 2$ and $T = 3$). On the other hand, it fully illustrates the features of solving optimization problems for multicriterial mathematical models with parameters.

Let the objective functions of the model to be maximized with respect to $X \in E^3$ with $V \in E^2$ be

$$F_1(X,V) = x_1\,, \qquad F_2(X,V) = x_2\,, \qquad F_3(X,V) = x_3\,.$$

The set of admissible states is defined by the system of inequalities

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \qquad \frac{x_1}{v_1} + \frac{x_2}{v_2} + \frac{x_3}{A - v_1 - v_2} \leq 1\,,$$

and the set $\Upsilon$ is defined as

$$1 \leq v_1 \leq B, \qquad 1 \leq v_2 \leq B, \qquad 2v_1 + v_2 \geq C\,,$$

where $A = 11,\ B = 5$ and $C = 7$.

In the model under consideration $K = 3$ and the *first level* problems will have the form:

$$\forall k = 1,2,3 \quad \textit{maximize over } x \in E^3 \qquad F_{(k)} = x_{(k)k}$$

*subject to*

$$x_{(k)1} \geq 0,\ x_{(k)2} \geq 0,\ x_{(k)3} \geq 0, \quad \frac{x_{(k)1}}{v_1} + \frac{x_{(k)2}}{v_2} + \frac{x_{(k)3}}{A - v_1 - v_2} \leq 1\,.$$

To solve these problems, we use a feedback function of the form

$$Q(\tau, s) = \frac{\tau}{2}\left(s - \frac{1}{s}\right) \qquad \text{c} \qquad R(\tau, s) = \frac{\tau}{2}\left(\frac{s^2}{2} - \ln s - \frac{1}{2}\right).$$

Then the auxiliary functions (3.4.6) for them are written as

$$U_{(k)}(X,V) =$$

$$= x_{(k)k} - \lambda_{(k)}\left(\frac{x_{(k)1}}{v_1} + \frac{x_{(k)2}}{v_2} + \frac{x_{(k)3}}{A - v_1 - v_2} - 1\right) - \qquad (3.4.16)$$

$$- R(\tau, x_{(k)1}) - R(\tau, x_{(k)2}) - R(\tau, x_{(k)3}) + R(\tau, \lambda_{(k)})\,.$$

Accordingly, the stationarity conditions of $U_{(k)}(X,V)$ for fixed $V \in \Upsilon$ will be

$$\left\{\begin{array}{rcl} \delta_{k1} - \dfrac{\overline{\lambda}_{(k)}}{v_1} - Q(\tau, \overline{x}_{(k)1}) &=& 0\,, \\ \delta_{k2} - \dfrac{\overline{\lambda}_{(k)}}{v_2} - Q(\tau, \overline{x}_{(k)2}) &=& 0\,, \\ \delta_{k3} - \dfrac{\overline{\lambda}_{(k)}}{A - v_1 - v_2} - Q(\tau, \overline{x}_{(k)3}) &=& 0\,, \\ \dfrac{\overline{x}_{(k)1}}{v_1} + \dfrac{\overline{x}_{(k)2}}{v_2} + \dfrac{\overline{x}_{(k)3}}{A - v_1 - v_2} - 1 - Q(\tau, \overline{\lambda}_{(k)}\ ) &=& 0\,. \end{array}\right. \tag{3.4.17}$$

In turn, the solutions of system (3.4.17) allow us to find the values of functions (3.4.8), which in the example under consideration are determined by the formulas following from (3.4.16) and (3.4.17),

$$\begin{aligned} \overline{U}_{(k)}(\tau, V) = x_{(k)k}(\tau, V) - \\ -R(\tau, \overline{x}_{(k)1}(\tau, V)) - R(\tau, \overline{x}_{(k)2}(\tau, V)) - R(\tau, \overline{x}_{(k)3}(\tau, V)) + \\ +R(\tau, \overline{\lambda}_{(k)}(\tau, V)) - \overline{\lambda}_{(k)}(\tau, V)\, Q(\tau, \overline{\lambda}_{(k)}(\tau, V))\,. \end{aligned} \tag{3.4.18}$$

The components of their gradients with respect to $V$ will be equal to $\forall k = \overline{1, K}$ .

$$\begin{array}{rcl} \dfrac{\partial \overline{U}_{(k)}}{\partial v_1} &=& \dfrac{\overline{\lambda}_{(k)} \overline{x}_{(k)1}}{v_1^2} - \dfrac{\overline{\lambda}_{(k)} \overline{x}_{(k)3}}{(A - v_1 - v_2)^2}\,, \\ \dfrac{\partial \overline{U}_{(k)}}{\partial v_2} &=& \dfrac{\overline{\lambda}_{(k)} \overline{x}_{(k)2}}{v_2^2} - \dfrac{\overline{\lambda}_{(k)} \overline{x}_{(k)3}}{(A - v_1 - v_2)^2}\,. \end{array} \tag{3.4.19}$$

We now apply the method of feedback functions to solve the problem *of the second level* (3.4.4).

In the formulation of this problem, instead of the dependencies $F^*_{(k)v}$ we use their smoothed approximations $\overline{U}_{(k)}(\tau, v_1, v_2) \quad \forall k = 1, 2, 3$ .

Then the second level problem for the model under consideration will take the form:

$$
\begin{aligned}
&\textit{maximize} \quad -\rho \\
&\textit{subject to} \;\; \rho \geq 0, \quad x_1 \geq 0,\; x_2 \geq 0,\; x_3 \geq 0\;, \\
&\qquad \frac{x_1}{v_1}+\frac{x_2}{v_2}+\frac{x_3}{A-v_1-v_2} \leq 1\;, \qquad (3.4.20) \\
&\qquad \overline{U}_{(1)}(\tau, v_1, v_2) - x_1 - \rho \leq 0\;, \\
&\qquad \overline{U}_{(2)}(\tau, v_1, v_2) - x_2 - \rho \leq 0\;, \\
&\qquad \overline{U}_{(3)}(\tau, v_1, v_2) - x_3 - \rho \leq 0\;.
\end{aligned}
$$

The auxiliary function of the second level problem will be

$$
\begin{aligned}
&\mathbb{U}\big(\tau, \rho, X, \Lambda, \mu\big) = \\
&\quad = -\rho - \lambda\left(\frac{x_1}{v_1}+\frac{x_2}{v_2}+\frac{x_3}{A-v_1-v_2}-1\right)- \\
&\quad -\mu_1\Big(\overline{U}_{(1)}(\tau, v_1, v_2) - x_1 - \rho\Big)- \\
&\quad -\mu_2\Big(\overline{U}_{(2)}(\tau, v_1, v_2) - x_2 - \rho\Big)- \\
&\quad -\mu_3\Big(\overline{U}_{(3)}(\tau, v_1, v_2) - x_3 - \rho\Big)- \\
&\quad -R(\tau,\rho) - R(\tau, x_1) - R(\tau, x_2) - R(\tau, x_3) + R(\tau, \lambda) + \\
&\quad +R(\tau, \mu_1) + R(\tau, \mu_2) + R(\tau, \mu_3)\;,
\end{aligned}
$$

where $\lambda$ and $\mu_1$, $\mu_2$, $\mu_3$ are the Lagrange multipliers of the last four constraints in (3.4.20).

We now write out the conditions that will determine the stationary points of the auxiliary function in the second-level problem (3.4.20).

$$
\left\{
\begin{array}{rcl}
\dfrac{\partial \mathbb{U}}{\partial \rho} &=& 0\,, \\
\dfrac{\partial \mathbb{U}}{\partial x_1} &=& 0\,, \\
\dfrac{\partial \mathbb{U}}{\partial x_2} &=& 0\,, \\
\dfrac{\partial \mathbb{U}}{\partial x_3} &=& 0\,, \\
\dfrac{\partial \mathbb{U}}{\partial \lambda} &=& 0\,, \\
\dfrac{\partial \mathbb{U}}{\partial \mu_1} &=& 0\,, \\
\dfrac{\partial \mathbb{U}}{\partial \mu_2} &=& 0\,, \\
\dfrac{\partial \mathbb{U}}{\partial \mu_3} &=& 0
\end{array}
\right.
\quad \text{or} \quad
\left\{
\begin{array}{rcl}
-1 + \overline{\overline{\mu}}_1 + \overline{\overline{\mu}}_2 + \overline{\overline{\mu}}_3 - Q(\tau, \overline{\overline{\rho}}) &=& 0\,, \\
-\dfrac{\overline{\overline{\lambda}}}{v_1} + \overline{\overline{\mu}}_1 - Q(\tau, \overline{\overline{x}}_1) &=& 0\,, \\
-\dfrac{\overline{\overline{\lambda}}}{v_2} + \overline{\overline{\mu}}_2 - Q(\tau, \overline{\overline{x}}_2) &=& 0\,, \\
-\dfrac{\overline{\overline{\lambda}}}{A - v_1 - v_2} + \overline{\overline{\mu}}_3 - Q(\tau, \overline{\overline{x}}_3) &=& 0\,, \\
\dfrac{\overline{\overline{x}}_1}{v_1} + \dfrac{\overline{\overline{x}}_2}{v_2} + \dfrac{\overline{\overline{x}}_3}{A - v_1 - v_2} - 1 - Q(\tau, \overline{\overline{\lambda}}) &=& 0\,, \\
\overline{U}_{(1)}(\tau, v_1, v_2) - \overline{\overline{x}}_1 - \overline{\overline{\rho}} - Q(\tau, \overline{\overline{\mu}}_1) &=& 0\,, \\
\overline{U}_{(2)}(\tau, v_1, v_2) - \overline{\overline{x}}_2 - \overline{\overline{\rho}} - Q(\tau, \overline{\overline{\mu}}_2) &=& 0\,, \\
\overline{U}_{(3)}(\tau, v_1, v_2) - \overline{\overline{x}}_3 - \overline{\overline{\rho}} - Q(\tau, \overline{\overline{\mu}}_3) &=& 0\,.
\end{array}
\right.
\tag{3.4.21}
$$

Having obtained the solutions of system (3.4.21), we can construct a smoothing approximation for the dependence $\rho_v^{**}$, which is used in the formulation of the third-level problem. As such an approximation we take the function

$$
\overline{\overline{\mathbb{U}}}(\tau, V) = \mathbb{U}\Big( \tau, \overline{\overline{\rho}}(\tau, V), \overline{\overline{x}}(\tau, V), \overline{\overline{\lambda}}(\tau, V), \overline{\overline{\mu}}(\tau, V) \Big),
$$

the components of whose gradient with respect to $V$ according to (3.4.14) are equal, respectively, to

$$
\begin{array}{rcl}
\dfrac{\partial \overline{\overline{\mathbb{U}}}}{\partial v_1} &=& \dfrac{\overline{\overline{\lambda}}\,\overline{\overline{x}}_1}{v_1^2} - \dfrac{\overline{\overline{\lambda}}\,\overline{\overline{x}}_3}{(A - v_1 - v_2)^2} - \overline{\overline{\mu}}_1 \dfrac{\partial \overline{U}_{(1)}}{\partial v_1} - \overline{\overline{\mu}}_2 \dfrac{\partial \overline{U}_{(2)}}{\partial v_1} - \overline{\overline{\mu}}_3 \dfrac{\partial \overline{U}_{(3)}}{\partial v_1}, \\
\text{and} & & \\
\dfrac{\partial \overline{\overline{\mathbb{U}}}}{\partial v_2} &=& \dfrac{\overline{\overline{\lambda}}\,\overline{\overline{x}}_2}{v_2^2} - \dfrac{\overline{\overline{\lambda}}\,\overline{\overline{x}}_3}{(A - v_1 - v_2)^2} - \overline{\overline{\mu}}_1 \dfrac{\partial \overline{U}_{(1)}}{\partial v_2} - \overline{\overline{\mu}}_2 \dfrac{\partial \overline{U}_{(2)}}{\partial v_2} - \overline{\overline{\mu}}_3 \dfrac{\partial \overline{U}_{(3)}}{\partial v_2},
\end{array}
\tag{3.4.22}
$$

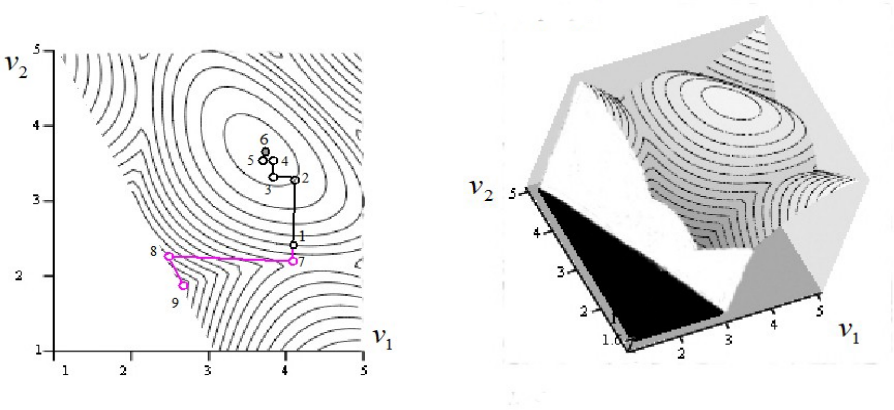



Fig. 3.4.1. System of isolines and 3d-graph of the function $-\overline{\overline{\mathbb{U}}}(\tau, v_1, v_2)$

where the values of the partial derivatives on the right-hand sides are determined by formulas (3.4.19).

For greater clarity, let us consider an easily verifiable version of the third-level problem:

*find local extreme values of the dependence* $\rho_v^{**}$ *in the case when the set* $\Upsilon \subset E^2$ *is determined by a system of inequalities*

$$\begin{cases} 1 \le v_1 \le 5\,, \\ 1 \le v_2 \le 5\,, \\ 2v_1 + v_2 \ge 7\,. \end{cases}$$

Geometrically, this problem allows for the following interpretation: it is required to find the values of the parameters $v_1$ and $v_2$, *optimizing* (by the magnitude of the discrepancy between the criteria) *the form* of the smoothed approximation of the Pareto set of the multicriteria model under consideration.

The system of isolines and the spatial graph approximation of the dependence $\rho_v^{**}$, that is, the functions $-\overline{\overline{\mathbb{U}}}(\tau, v_1, v_2)$, are shown in Fig. 3.4.1.

To find the maximum in the third-level problem, the *steepest ascent* method was used. The *normalized gradient* of the function $-\overline{\overline{\mathbb{U}}}(\tau, v_1, v_2)$ was used as the direction vector $w$ . The step size estimate was determined by the values of this function along $w$ .

The main quantitative characteristics for several initial iterations of the solution process at $\tau = 0.01$ are given in Table 3.4.1a—3.4.1b. The exact solution of the third level problem (search for a local maximum) (point 6 in Fig. 3.4.1) has the form:

$$v_1^{**} = \frac{11}{3}\,, \quad v_2^{**} = \frac{11}{3}\,, \quad \rho_{v^{**}}^{**} = \frac{22}{9}\,.$$

T a b l e 3,4.1a.

| Point in Fig. 4 | $v_1$ | $v_2$ | $-\overline{\overline{U}}$ |
|---|---|---|---|
| 1 | 4.000000000 | 2.500000000 | 2.292876919 |
| 2 | 4.108927815 | 3.418563950 | 2.427442369 |
| 3 | 3.751137224 | 3.458387226 | 2.440214361 |
| 4 | 3.766947377 | 3.609311384 | 2.443407623 |
| 5 | 3.686757173 | 3.617925071 | 2.444096417 |
| *Exact solution* | 3.666666667 | 3.666666667 | 2.4444444444 |

The local minimum in the third-level problem was determined by the *antigradient projection* method. The quantitative characteristics of the first three steps of the corresponding iterative process for $\tau = 0.01$ are given in Table 3.4.2a—3.4.2b.

The exact solution of this problem (point 9 in Fig. 3.4.1) has the form:

$$v_1^{**} = \frac{1+\sqrt{141}}{5}\,, \quad v_2^{**} = \frac{33-2\sqrt{141}}{5}\,, \quad \rho_{v^{**}}^{**} = \frac{33-2\sqrt{141}}{5}\,.$$

To demonstrate the smoothing property of the feedback function method, Fig. 3.4.2 shows the graphs of the function

$$\overline{\Phi}(\tau, v_1) = -\overline{\overline{\mathbb{U}}}(\tau, v_1, 7 - 2v_1,)$$

for $\tau = 0.05$, 0.025, 0.01, as well as for $\tau \to +0$.

T a b l e 3.4.1b.

| Point in Fig.4 | Standard deg. | $w_1$ | $w_2$ | Step |
|---|---|---|---|---|
| 1 | 0.292812466 | 0.117759800 | 0.993042109 | 0.925000000 |
| 2 | 0.070259924 | −0.993862751 | 0.110620212 | 0.360000000 |
| 3 | 0.041937320 | 0.104185521 | 0.994557880 | 0.151750000 |
| 4 | 0.017053825 | −0.994280382 | 0.106801322 | 0.080651500 |
| 5 | 0.009512746 | ... | ... | ... |

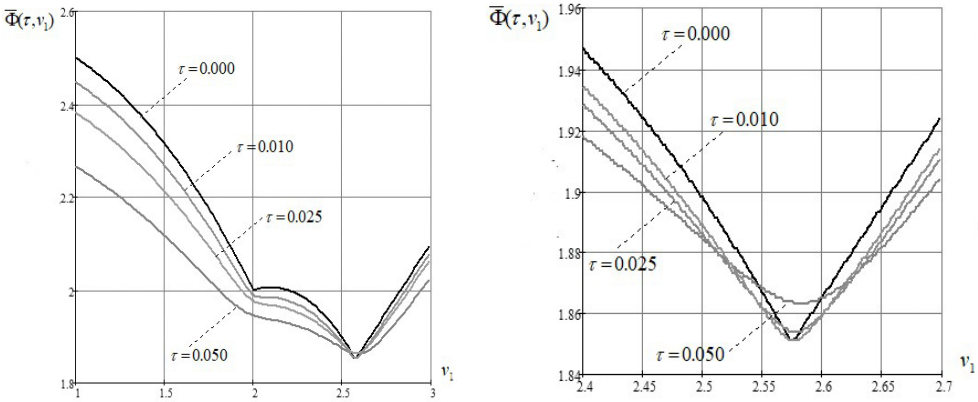



Fig. 3.4.2. Function graphs $\overline{\Phi}(\tau, v_1)$

T a b l e 3.4.2a.

| Point in Fig.4 | $v_1$ | $v_2$ | $-\overline{\overline{U}}$ |
|---|---|---|---|
| 1 | 4.000000000 | 2.500000000 | 2.292876919 |
| 7 | 3.962274029 | 2.181865306 | 2.181828016 |
| 8 | 2.311246750 | 2.377506503 | 1.976736712 |
| 9 | 2.574871109 | 1.850257785 | 1.850234349 |
| *Exact solution* | 2.574868417 | 1.850263165 | 1.850263165 |

T a b l e 3.4.2b.

| Point in Fig.4 | Standard deg. | $w_1$ | $w_2$ | Step |
|---|---|---|---|---|
| 1 | 0.292812466 | −0.117759800 | −0.993042109 | 0.320363750 |
| 7 | 0.085751139 | −0.993052354 | 0.117673374 | 1.662578285 |
| 8 | 0.433886194 | 0.447213595 | −0.894427191 | 0.589481987 |
| 9 | ... | ... | ... | ... |

## 3.5. Game theory problems

Another area of operations research, where superpositions of extremum search operations are used, is game theory.

Let us consider the application of the feedback function method to solve a simple game problem. In this case, we will slightly change its condition to fully demonstrate the features of using this algorithm.

Let two players participate in the game: the first $A$ and the second $B$. Both make only one move. A move consists of choosing one from a pre-known set of actions, which we will call *strategy*. In this case, the players do not know the opponent's choice. For each pair of selected strategies, the *outcome of the game* is known: the amount of one player's gain, equal to the loss of the other.

The problem requires finding a choice of strategies that maximizes the guaranteed gain for player $A$.

Let us consider the following variant of its formulation.

Let players $A$ and $B$ have sets of strategies with numbers $j = \overline{1,n}$ and $i = \overline{1,m}$ respectively.

The outcome of the game is a real number $\alpha_{ij}$, equal to the gain of player $A$ (and, accordingly, the loss of $B$) when the first player chooses the strategy with number $j$, and the second player chooses the strategy with number $i$. The outcomes of the game can be written in the form of a matrix $\|\alpha_{ij}\|$, which we will call the *payoff matrix*.

It is easy to conclude that the optimal strategy of player $A$ is to choose $j^+$ and $i^+$ such that

$$\alpha_{j^+i^+} = \max_{j=\overline{1,n}} \min_{i=\overline{1,m}} \alpha_{ij} \,.$$

Similarly, we can formulate the problem of finding the optimal strategy of player $B$. For example, find a strategy that minimizes his inevitable loss, that is, find $j^\times$ and $i^\times$ for which

$$\alpha_{j^\times i^\times} = \min_{i=\overline{1,m}} \max_{j=\overline{1,n}} \alpha_{ij} \,.$$

In this case, the question naturally arises: do the optimal strategies of players $A$ and $B$ lead to the same outcome of the game? That is, is the equality

$$\alpha_{j^+i^+} = \alpha_{j^\times i^\times}$$

true? This is possible. Let's say, in the case when all elements in the payoff matrix are the same. However, the example in §3.3 shows that this equality may not hold.

Let us consider this game problem in more detail, slightly complicating its formulation.

First, we will assume that the elements of the payoff matrix are sufficiently smooth functions of the vector of exogenous parameters

$$V = \| v_1, v_2, \dots v_K \|^{\mathrm{T}},$$

belonging to a certain domain $\Upsilon \subseteq E^K$. That is, the payoff matrix is *a matrix function* of the form $\| \alpha_{ij}(V) \|$.

Second, we assume that the choice of strategy in the game is performed *an unlimited* number of times. Moreover, the conditions *of all choices are the same and do not depend on the results of others.*

Finally, let each player choose one of the strategies available to him *randomly.* We will assume that the first player chooses strategy $j$ with probability $x_j$. It is clear that in this case

$$x_j \geq 0 \quad j = \overline{1,n} \qquad \text{and} \qquad \sum_{j=1}^{n} x_j = 1\,.$$

The vector $X = \| x_1, x_2, \dots x_n \|^{\mathrm{T}}$ is usually called the *mixed strategy* of player $A$.

We also note that the solution to the original problem with a *single* choice of strategies will be a mixed strategy, all components of which are equal to zero, except a single, which equals to one. Such solutions, called solutions *in pure strategies*, as we have seen, do not exist for every payoff matrix.

It is easy to see that the problem of finding the optimal mixed strategy of the first player is a problem of linear parametric programming:

$$\begin{aligned} &\textit{maximize with respect to } \{x, f\} \quad \textit{function } f \\ &\textit{subject to} \quad x_j \geq 0 \quad j = \overline{1,n}, \qquad \sum_{j=1}^{n} x_j = 1\,, \\ &\qquad\qquad \sum_{j=1}^{n} \alpha_{ij}(V)\, x_j \geq f \qquad i = \overline{1,m}\,. \end{aligned} \tag{3.5.1}$$

We now formulate the problem of finding an optimal mixed strategy for player $B$.

Let the second player choose strategy $i$ with probability $\lambda_i$ and vector $\Lambda = \| \lambda_1, \lambda_2, \dots \lambda_m \|^{\mathrm{T}}$. Then we get the problem

$$\begin{aligned} &\textit{minimize by } \{\Lambda, g\} \quad \textit{function } g \\ &\textit{under conditions} \quad \lambda_i \geq 0 \quad i = \overline{1,m}, \qquad \sum_{i=1}^{m} \lambda_i = 1\,, \\ &\qquad\qquad \sum_{i=1}^{m} \alpha_{ij}(V)\,\lambda_i \leq g \qquad j = \overline{1,n}\,. \end{aligned} \tag{3.5.2}$$

This problem is dual to problem (3.5.1).

In problem (3.5.2), the quantity $\sum\limits_{i=1}^{m} \alpha_{ij}(V)\lambda_i$ is *the mathematical expectation* of the loss of player $B$ when player $A$ chooses the $j$-th strategy. We will denote the solution of problem (3.5.2) by $\{\Lambda_v^*,\, g_v^*\}$. In this case, the value $\sum\limits_{j=1}^{n} \alpha_{ij}(V)\, x_j$ turns out to be the *mathematical expectation* of the payoff of player $B$ when choosing the $i$-th strategy.

It is known from the theory of linear programming that for dual pairs of problems in the case of bounded solutions the values of the primary and dual objective functions are equal. This means that in mixed strategies, the optimal choices of players $A$ and $B$ always lead to the same outcome of the game.

Let us find out how the optimal strategies of the players will depend on the vector of exogenous parameters $V$.

As noted, the solution to the problem (3.5.1) is the dependence $\{X_v^*,\, f_v^*\}$, which may not be defined in all $E^n$, not be a function of $V$, and in the case of functionality, not have derivatives with respect to the components of $V$. To overcome these difficulties, we use the smoothing property of the method of feedback functions.

Let the components of the vector $V$ and the parameter $\tau > 0$ be fixed. Then the use of the method of feedback functions for solving the pair of problems (3.5.1) — (3.5.2) is reduced to finding stationary points by $\{X, \Lambda, f, g\}$ for the auxiliary function

$$U(\tau, X, \Lambda, f, g, V) =$$

$$= f - \sum_{i=1}^{m} \lambda_i \Big( f - \sum_{j=1}^{n} \alpha_{ij}(V)\, x_j \Big) - g \Big( -1 + \sum_{j=1}^{n} x_j \Big) - \sum_{j=1}^{n} R\big(\tau, x_j\big) + \sum_{i=1}^{m} R\big(\tau, \lambda_i\big). \tag{3.5.3}$$

These stationarity conditions are determined by a system of equations of the form

$$\left\{\begin{array}{rcl} \sum\limits_{i=1}^{m}\alpha_{ij}(V)\,\overline{\lambda}_i-\overline{g}-Q\big(\tau,\overline{x}_j\big) &=& 0 \qquad j=\overline{1,n}\,,\\ 1-\sum\limits_{i=1}^{m}\overline{\lambda}_j=0\,, \qquad 1-\sum\limits_{j=1}^{n}\overline{x}_j &=& 0\,,\\ \sum\limits_{j=1}^{n}\alpha_{ij}(V)\,\overline{x}_j-\overline{f}+Q\big(\tau,\overline{\lambda}_i\big) &=& 0 \qquad i=\overline{1,m}\,. \end{array}\right. \tag{3.5.4}$$

In Theorem 2.2.2 it is proved that $\forall V\in E^K$ a system of the form (3.5.4) is uniquely solvable. Moreover, for its solutions $\overline{f}(\tau,V)$ and $\overline{g}(\tau,V)$ the equalities

$$\lim_{\tau\to+0}\overline{f}(\tau.V)=f_v^* \qquad \text{and} \qquad \lim_{\tau\to+0}\overline{g}(\tau.V)=g_v^*\,.$$

Then, by virtue of Theorem 2.3.3, the functions $\overline{f}(\tau,V)$ and $\overline{g}(\tau,V)$ can be used as smooth approximations of the dependencies $f_v^*$ and $g_v^*$.

Let the second partial derivatives of all functions $\alpha_{ij}(V)$ be continuous. In this case, to solve problems, the formulation of which includes the dependencies $f_v^*$ and $g_v^*$, one can use Taylor approximations of the functions $\overline{f}(\tau,V)$ and $\overline{g}(\tau,V)$ up to the second order inclusive.

Finally, the combined use of the rule of differentiation of a composite function and the theorem on implicit functions for conditions (3.5.4) allows us to adopt as an approximation of the gradient of the function $\overline{f}(\tau,V)$ a vector function with components

$$\overline{U}'_{v_t}(\tau,V)=\frac{\partial U}{\partial v_t}\Big(\tau,\overline{X},\overline{\Lambda},\overline{f},\overline{g},V\Big)=\sum_{i=1}^{m}\sum_{j=1}^{n}\overline{x}_j\overline{\lambda}_i\frac{\partial\alpha_{ij}(V)}{\partial v_t} \qquad t=\overline{1,K}\,. \tag{3.5.5}$$

As an example, we use the method of feedback functions to study solutions to the game problem with a payoff matrix considered in §3.3 and depending on two scalar parameters $v_1$ and $v_2$.

$$\|D_{ij}(V)\|=\left\|\begin{array}{cccc} 1 & 25-v_1-v_2 & 3 & 4\\ 5 & v_1 & 7 & 8\\ 9 & v_2 & 11 & 12 \end{array}\right\|. \tag{3.5.6}$$

It is easy to see that for this matrix the quantities

$$D_{\text{minmax}}=\min_{i=\overline{1,3}}\Big(\max_{j=\overline{1,4}}D_{ij}(v_1,v_2)\Big) \quad \text{and} \quad D_{\text{maxmin}}=\max_{j=\overline{1,4}}\Big(\min_{i=\overline{1,3}}D_{ij}(v_1,v_2)\Big)$$

are continuous, piecewise linear functions of $v_1$ and $v_2$. Let us assume that the values of their arguments $v_1 \in [0, 15]$ and $v_2 \in [0, 15]$.

To study the properties of these functions, we use their smooth approximations, which are obtained by the method of feedback functions from formulas (3.3.2) and (3.3.3) and have the form:

$$\overline{D}_{\text{minmax}}(\tau, v_1, v_2) = -\tau \ln \sum_{j=1}^{4} \left( \sum_{i=1}^{3} e^{\frac{D_{ij}(v_1, v_2)}{\tau}} \right)^{-1}$$

$$\overline{D}_{\text{maxmin}}(\tau, v_1, v_2) = \tau \ln \sum_{i=1}^{3} \left( \sum_{j=1}^{4} e^{-\frac{D_{ij}(v_1, v_2)}{\tau}} \right)^{-1} . \tag{3.5.7}$$

The isolines of the functions $\overline{D}_{\text{minmax}}(\tau, v_1, v_2)$ and $\overline{D}_{\text{maxmin}}(\tau, v_1, v_2)$ are shown in Fig. 3.3.1 and 3.3.2, respectively for the value $\tau = 0.05$. These figures also show their values at characteristic points.

It is known that a solution to the pair of problems (3.5.1)—(3.5.2) in mixed strategies exists for any payoff matrix. However, solutions in *pure strategies* are of particular interest. That is, solutions in which all components of both the vector $x$ and the vector $L$ are equal to zero, except for one, which is equal to one.

The condition for the existence of such a solution is the presence of a *saddle element* in the payoff matrix, that is, an element $\alpha_{ij}$ for which

$$\alpha_{ij} = D_{\text{minmax}} = D_{\text{maxmin}} .$$

The fulfillment of this condition obviously depends on the values of the parameters $v_1$ and $v_2$. Figure 3.5.1 shows a system of isolines for the function

$$\overline{D}_{\text{minmax}}(\tau, v_1, v_2) - \overline{D}_{\text{maxmin}}(\tau, v_1, v_2)$$

for the value $\tau = 0.05$. Here, the points at which the matrix (3.5.6) has saddle elements are marked in black.

As smoothed approximations of the dependencies $D_{\text{minmax}}$ and $D_{\text{maxmin}}$ on $\tau$, $v_1$ and $v_2$, we can use the solutions of system (3.5.4) and formulas (3.5.5). Note also that in the example under consideration, the derivatives (3.5.5) are found using the formulas

$$\overline{U}'_{v_1} = -\overline{x}_2\overline{\lambda}_1 + \overline{x}_2\overline{\lambda}_2 \qquad \text{and} \qquad \overline{U}'_{v_2} = -\overline{x}_2\overline{\lambda}_1 + \overline{x}_2\overline{\lambda}_3 .$$

As an illustration, Tables 3.5.1$a$ and 3.5.1$b$ show the values of the arguments of the stationary points of the auxiliary function (3.5.3) for $v_1 = 10$ and $v_2 = 9$ for different $\tau$.

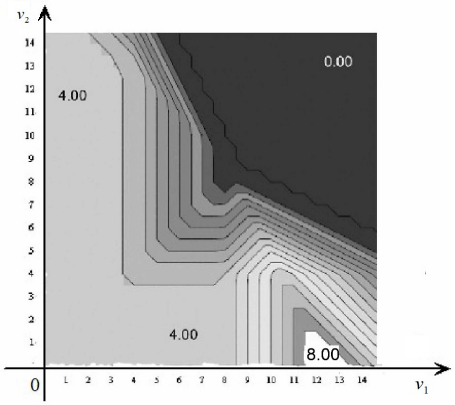



Fig. 3.5.1. System of isolines of the function $\overline{D}_{\text{minmax}}(\tau, v_1, v_2) - \overline{D}_{\text{maxmin}}(\tau, v_1, v_2)$

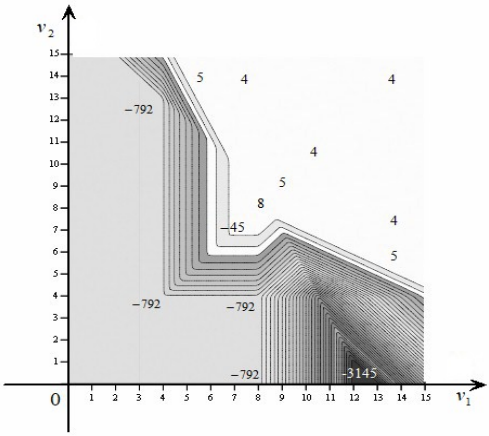



Fig. 3.5.2. System of isolines of the function $\Phi(\Theta, v_1, v_2)$ in scale 1.00

T a b l e 3.5.1a

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{x}_3$ | $\overline{x}_4$ | $\overline{f}$ |
|---|---|---|---|---|---|
| $10^{-1}$ | 0.010143028 | 0.946888255 | 0.017065774 | 0.025902943 | 5.843463575 |
| $10^{-2}$ | $1.0016 \cdot 10^{-3}$ | 0.994816832 | $1.6712 \cdot 10^{-3}$ | $2.5103 \cdot 10^{-3}$ | 5.984928431 |
| $10^{-3}$ | $1.0002 \cdot 10^{-4}$ | 0.999483166 | $1.6671 \cdot 10^{-4}$ | $2.5010 \cdot 10^{-4}$ | 5.998499278 |
| $10^{-4}$ | $1.0000 \cdot 10^{-5}$ | 0.999948332 | $1.6667 \cdot 10^{-5}$ | $2.5001 \cdot 10^{-5}$ | 5.999849993 |
| $10^{-5}$ | $1.0000 \cdot 10^{-6}$ | 0.999994833 | $1.6667 \cdot 10^{-6}$ | $2.5000 \cdot 10^{-6}$ | 5.999985000 |
| $10^{-6}$ | $1.0000 \cdot 10^{-7}$ | 0.999999483 | $1.6667 \cdot 10^{-7}$ | $2.5000 \cdot 10^{-7}$ | 5.999998500 |
| $10^{-7}$ | $1.0000 \cdot 10^{-8}$ | 0.999999948 | $1.6667 \cdot 10^{-8}$ | $2.5000 \cdot 10^{-8}$ | 5.999999850 |
| $10^{-8}$ | $1.0000 \cdot 10^{-9}$ | 0.999999995 | $1.6667 \cdot 10^{-9}$ | $2.5000 \cdot 10^{-9}$ | 5.999999985 |
| *Exact solution* | 0 | 1 | 0 | 0 | 6 |

The values of the parameters for which saddle elements exist are not unique.

This allows us to expand the range of problem statements for game mathematical models. For example, we can consider the following problem.

T a b l e 3.5.1b

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{x}_3$ | $\overline{g}$ | $\overline{U}'_{v_1}$ | $\overline{U}'_{v_2}$ |
|---|---|---|---|---|---|---|
| $10^{-1}$ | 0.972216214 | 0.012489251 | 0.015294535 | 6.101300739 | −0.908754189 | −0.908754189 |
| $10^{-2}$ | 0.997097639 | $1.2500 \cdot 10^{-3}$ | $1.6524 \cdot 10^{-3}$ | 6.010009038 | −0.990686004 | −0.990285707 |
| $10^{-3}$ | 0.999708477 | $1.2500 \cdot 10^{-4}$ | $1.6652 \cdot 10^{-4}$ | 6.001000087 | −0.999066859 | −0.999025357 |
| $10^{-4}$ | 0.999970835 | $1.2500 \cdot 10^{-5}$ | $1.6665 \cdot 10^{-5}$ | 6.000100001 | −0.999906669 | −0.999902504 |
| $10^{-5}$ | 0.999997083 | $1.2500 \cdot 10^{-6}$ | $1.6667 \cdot 10^{-6}$ | 6.000010000 | −0.999990667 | −0.999990250 |
| $10^{-6}$ | 0.999999708 | $1.2500 \cdot 10^{-7}$ | $1.6667 \cdot 10^{-7}$ | 6.000001000 | −0.999999067 | −0.999999025 |
| $10^{-7}$ | 0.999999971 | $1.2500 \cdot 10^{-8}$ | $1.6667 \cdot 10^{-8}$ | 6.000000100 | −0.999999907 | −0.999999903 |
| $10^{-8}$ | 0.999999997 | $1.2500 \cdot 10^{-9}$ | $1.6667 \cdot 10^{-9}$ | 6.000000010 | −0.999999991 | −0.999999990 |
| *Точн. реш.* | 1 | 0 | 0 | 6 | −1 | −1 |

`Example 3.5.1.` For a game with payoff matrix (3.5.6) find the values of the parameters $v_1$ and $v_2$ for which the value $D_{\text{minmax}}$ has a maximum on the set of pure strategies.

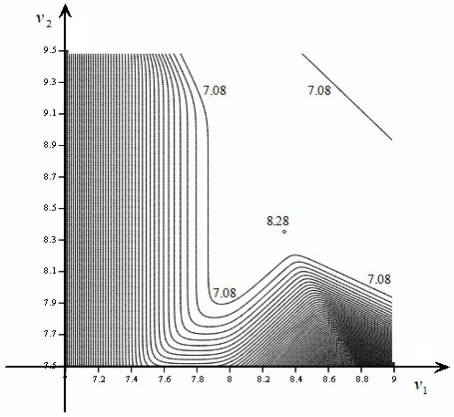



Fig. 3.5.3. System of function isolines $\Phi(\Theta, v_1, v_2)$ in scale 0.1

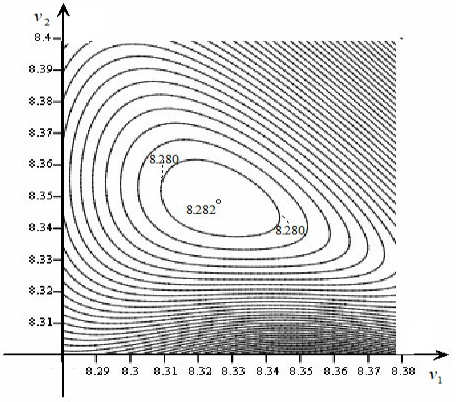



Fig. 3.5.4. System of isolines of the function $\Phi(\Theta, v_1, v_2)$ in scale 0.01

`Solution.` 1°. Since the existence of a solution in pure strategies requires the existence of a saddle point for the payoff matrix, the formulation of this problem can be as follows:

*maximize with respect to* $\{v_1, v_2\}$ *value* $D_{\text{minmax}}$

$$\text{subject to} \quad 0 \le v_1 \le 15\,, \quad 0 \le v_2 \le 15\,, \qquad (3.5.8)$$

$$D_{\text{minmax}} - D_{\text{maxmin}} = 0\,.$$

2°. Now we will use the method *penalty functions* [3, 29] to approximately estimate the desired values $v_1$ and $v_2$. The application of this method in the example under consideration can be reduced to the unconditional maximization of an auxiliary function of the form:

$$\Phi(\Theta, v_1, v_2) = \overline{D}_{\text{minmax}}(v_1, v_2) -$$

$$-\frac{1}{2\Theta}\Big(\overline{D}_{\text{minmax}}(v_1, v_2) - \overline{D}_{\text{maxmin}}(v_1, v_2)\Big)^2\,, \quad (3.5.9)$$

for a small positive value of $\Theta$.

The function $\Phi(\Theta, v_1, v_2)$ is a smoothed approximation of the objective function of the problem (3.5.8). The smoothing effect is illustrated by Fig. 3.5.2 — 3.5.4, which show isolines for the function (3.5.9) in different scales and give its values at some points. Here, the value $\Theta = 0.01$ was used.

The maximum of the function $\Phi(\Theta, v_1, v_2)$ at $\Theta = 0.01$ is achieved at the point for which $\overline{v}_1 = 8.325815949$, $\overline{v}_2 = 8.348452929$ and $\Phi(\Theta, \overline{p}, \overline{q}) = 8.28158929$. The norm of the gradient of the function $\Phi(\Theta, v_1, v_2)$ at this point is equal to $1.4541 \cdot 10^{-6}$.

T a b l e 3.5.2a

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{x}_3$ | $\overline{x}_4$ | $\overline{f}$ |
|---|---|---|---|---|---|
| $10^{-1}$ | $9.1459 \cdot 10^{-3}$ | 0.956169686 | 0.014420881 | 0.020263533 | 8.058310734 |
| $10^{-2}$ | $9.1515 \cdot 10^{-4}$ | 0.995611686 | $1.4436 \cdot 10^{-3}$ | $2.0296 \cdot 10^{-3}$ | 8.305809423 |
| $10^{-3}$ | $9.1516 \cdot 10^{-5}$ | 0.999561163 | 0.000144361 | $2.0296 \cdot 10^{-4}$ | 8.330580921 |
| $10^{-4}$ | $9.1516 \cdot 10^{-6}$ | 0.999956116 | $1.4436 \cdot 10^{-5}$ | $2.0296 \cdot 10^{-5}$ | 8.333058092 |
| $10^{-5}$ | $9.2730 \cdot 10^{-7}$ | 0.999995508 | $1.4740 \cdot 10^{-6}$ | $2.0903 \cdot 10^{-6}$ | 8.333306242 |
| $10^{-6}$ | $1.3369 \cdot 10^{-7}$ | 0.999998903 | $2.8735 \cdot 10^{-7}$ | $6.7565 \cdot 10^{-7}$ | 8.333330906 |
| $10^{-7}$ | $1.5512 \cdot 10^{-8}$ | 0.999999720 | $4.0876 \cdot 10^{-8}$ | $2.2400 \cdot 10^{-7}$ | 8.333333009 |
| *Точн. реш.* | 0 | 1 | 0 | 0 | $\frac{25}{3}$ |

T a b l e 3.5.2b

| $\tau$ | $\overline{\lambda}_1$ | $\overline{\lambda}_2$ | $\overline{\lambda}_3$ | $\overline{g}$ | $\overline{U}'_{v_1}$ | $\overline{U}'_{v_2}$ |
|---|---|---|---|---|---|---|
| $10^{-1}$ | 0.657125682 | 0.217912820 | 0.124961498 | 8.337816823 | −0.419962024 | −0.508839261 |
| $10^{-2}$ | 0.657393186 | 0.217761533 | 0.124845281 | 8.333377313 | −0.437702412 | −0.530210918 |
| $10^{-3}$ | 0.657395820 | 0.217760043 | 0.124844137 | 8.333333772 | −0.439442849 | −0.532317980 |
| $10^{-4}$ | 0.657395847 | 0.217760028 | 0.124844125 | 8.333333338 | −0.439616526 | −0.532528351 |
| $10^{-5}$ | 0.640002794 | 0.234666153 | 0.125331054 | 8.333333333 | −0.405334820 | −0.514669428 |
| $10^{-6}$ | 0.410689714 | 0.280293397 | 0.309016889 | 8.333333334 | −0.130396173 | −0.101672713 |
| $10^{-7}$ | 0.257461958 | 0.457547439 | 0.284990604 | 8.333333333 | 0.200085425 | 0.027528638 |
| *Точн. реш.* | * * * | * * * | * * * | $\frac{25}{3}$ | * * * | * * * |

3°. To obtain an exact solution in the example under consideration, it is more convenient to use the method of linear extrapolation. Since this method is implemented by formulas similar to those considered earlier (2.4.3), here we will limit ourselves to presenting only the final estimate, which yields $v_1^* = 8.333333333$ and $v_2^* = 8.333333333$. Tables 3.5.2*a* and 3.5.2*b* show the solution of system (3.5.4) for these values of $v_1$ and $v_2$ for different values of the parameter $\tau$.

Note that for optimal values of the parameters, the saddle elements of the matrix (3.5.6) are the elements $D_{12}$, $D_{22}$ and $D_{32}$. The exact solutions of problems (3.5.1) — (3.5.2) are uniquely determined only for the variables $x_1, x_2, x_3, x_4, f$ and $g$.

This solution is not a solution in pure strategies for problem (3.5.2). The fact is that the solution of problem (3.5.1) for optimal values of $v_1^*$ and $v_2^*$, *is overdetermined*. That is, the number of active constraints in the optimum of problem (3.5.1) is greater than its dimension.

The solution to problem (3.5.2) in this case is not unique. The optimal values of the variables are an arbitrary set of non-negative numbers $\lambda_1^*$, $\lambda_2^*$, $\lambda_3^*$ such that $\lambda_1^* + \lambda_2^* + \lambda_3^* = 1$.

Among these solutions, there are three in pure strategies. One such solution with $\lambda_1^* = 0$, $\lambda_2^* = 1$, $\lambda_3^* = 0$ is shown in Table. 3.5.3*a*, 3.5.3*b* and 3.5.3*c*. It can be obtained, for example, by adding to system (3.5.4) the inequality $\lambda_2 \geq 1$. It has the same values of the objective functions as the solution in Tables 3.5.2*a* and 3.5.2*b*.

Other solutions in pure strategies, where $\lambda_1^* = 1$, $\lambda_2^* = 0$, $\lambda_3^* = 0$ and $\lambda_1^* = 0$, $\lambda_2^* = 0$, $\lambda_3^* = 1$ are found similarly.

T a b l e 3.5.3a

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{x}_3$ | $\overline{x}_4$ | $\overline{f}$ |
|---|---|---|---|---|---|
| $10^{-15}$ | 0 | 1 | 0 | $1.4500^{-15}$ | 8.333333314 |
| *Exact. solve* | 0 | 1 | 0 | 0 | $\frac{25}{3}$ |

T a b l e 3.5.3b

| $\tau$ | $\overline{\lambda}_1$ | $\overline{\lambda}_2$ | $\overline{\lambda}_3$ |
|---|---|---|---|
| $10^{-15}$ | $1.6928 \cdot 10^{-8}$ | 0.999999979 | $1.7177 \cdot 10^{-8}$ |
| *Exact. solved* | * * * | * * * | * * * |

T a b l e 3.5.3c

| $\tau$ | $\overline{g}$ | $\overline{U}'_{v_1}$ | $\overline{U}'_{v_2}$ |
|---|---|---|---|
| $10^{-15}$ | 8.333333445 | 0.999999962 | $2.4870 \cdot 10^{-10}$ |
| *Exact. solved* | $\frac{25}{3}$ | * * * | * * * |

# Chapter 4.

# Development of the method of feedback functions

Let us now consider some possible primaryions of further development of the method of feedback functions.

## 4.1. Analysis of the convergence of the method

The first of these areas includes increasing the efficiency of the practical use of the method of feedback functions.

It is of interest to analyze the dependence of the convergence rate of this method on the degree of conditionality of the problem being solved.

The reasoning turns out to be most illustrative in the linear case, when only one of the problems in the dual pair is poorly conditioned. Therefore, we recall

| | |
|---|---|
| `Definition 4.1.1` | A pair of mutually dual problems (1.2.1)—(1.2.2) is called *regular* if each of them<br>has a finite solution and, moreover, a unique one.<br>If at least one of the mutually dual problems is incompatible, we will call such a pair of problems *improper*.<br>A compatible problem, the number of active constraints of which at the optimal point exceeds the dimension of the problem, we will call *overdetermined*. |

It follows from the theorems proved in § 2.2 that system (1.2.7) has, for a fixed $\tau > 0$, a unique solution, regardless of whether problems (1.2.1) and (1.2.2) are regular or not. Moreover, in the regular case, for sufficiently small positive values of $\tau$, the solution of system (1.2.7) is close to the optimal points of problems (1.2.1)—(1.2.2).

Let us compare the results of applying different forms of the feedback function method to solving an irregular problem. Let us consider

`Example 4.1.1` *primary problem*:
maximize in $E^2$ the objective function

$$F = 2x_1 + 6x_2$$

subject to the conditions: $x_1 \geq 0\ ,\ x_2 \geq 0$ and

$$x_1 + 3x_2 \leq 3\ ,$$
$$2x_1 + x_2 \leq 6\ .$$

*Dual problem*:
minimize in $E^2$ the function

$$G = 3\lambda_1 + 6\lambda_2$$

subject to: $\lambda_1 \geq 0\ ,\ \lambda_2 \geq 0$ and

$$\lambda_1 + 2\lambda_2 \geq 2\ ,$$
$$3\lambda_1 + \lambda_2 \geq 6\ .$$

`Solution.` It is easy to see that both of these problems are *compatible*. In this case, the primary problem has *non-unique* solution $x_1^* = 3 - 3t,\ x_2^* = t \quad \forall t \in [0,1]$ with overdetermined point $\| \ 3 \ 0 \ \|^{\mathrm{T}}$.
The dual problem has a unique solution and *overdetermined* $\lambda_1^* = 2,\ \lambda_2^* = 0$, and $F^* = G^* = 6$.
Let in this problem $Q(\tau, s) = \dfrac{\tau}{2}\left(s - \dfrac{1}{s}\right)$, for which

$$\frac{\partial P}{\partial s} = \sqrt{\left(\frac{s}{\tau}\right)^2 + 1} + \frac{s}{\tau}\,.$$

Then the system (1.2.7) will have the following form:

$$\begin{cases} 3-\overline{x}_1-3\overline{x}_2 &= -\dfrac{\tau}{2}\left(\overline{\lambda}_1-\dfrac{1}{\overline{\lambda}_1}\right), \\ 6-2\overline{x}_1-\overline{x}_2 &= -\dfrac{\tau}{2}\left(\overline{\lambda}_2-\dfrac{1}{\overline{\lambda}_2}\right), \\ -2+\overline{\lambda}_1+2\overline{\lambda}_2 &= -\dfrac{\tau}{2}\left(\overline{x}_1-\dfrac{1}{\overline{x}_1}\right), \\ -6+3\overline{\lambda}_1+\overline{\lambda}_2 &= -\dfrac{\tau}{2}\left(\overline{x}_2-\dfrac{1}{\overline{x}_2}\right). \end{cases} \tag{4.1.1}$$

`The solution is obtained.` Its solutions are given in Table 4.1.1.

It can be noted that the solutions of system (4.1.1) are noticeably different from each other for different values of the parameter $\tau$. This is a consequence of the non-uniqueness of the solution to the primary problem. However, all solutions slightly violate the equality $\overline{x}_1+3\overline{x}_2=3$. This ensures that the error in the values of the objective function is small.

T a b l e 4.1.1

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{\lambda}_1$ | $\overline{\lambda}_2$ | $\lvert\overline{f}_1\rvert$ | $\overline{F}$ | $\overline{G}$ |
|---|---|---|---|---|---|---|---|
| $10^{-1}$ | 0.739168513 | 0.778693237 | 2.003973161 | 0.013355994 | 0.075248224 | 6.150496448 | 6.092055445 |
| $10^{-2}$ | 0.733869336 | 0.757877918 | 2.000494421 | 0.001324717 | 0.007503090 | 6.015006180 | 6.009431566 |
| $10^{-3}$ | 0.733333935 | 0.755805365 | 2.000050427 | 0.000132362 | 0.000750032 | 6.001500063 | 6.000945452 |
| $10^{-4}$ | 0.733280343 | 0.755598219 | 2.000005053 | 1.32351e-05 | 7.50003e-05 | 6.000150001 | 6.000094568 |
| $10^{-5}$ | 1.745625806 | 0.418127231 | 2.000001178 | 2.39163e-06 | 7.49988e-06 | 6.000015000 | 6.000017884 |
| $10^{-6}$ | 2.671672415 | 0.109442778 | 2.000000782 | 9.13722e-07 | 7.49977e-07 | 6.000001500 | 6.000007830 |
| $10^{-7}$ | 2.899707247 | 0.033430943 | 2.000000286 | 2.99124e-07 | 7.49940e-08 | 6.000000150 | 6.000002652 |
| $10^{-8}$ | 2.968350634 | 0.010549791 | 2.000000093 | 9.47886e-08 | 7.49823e-09 | 6.000000015 | 6.000000849 |
| $10^{-9}$ | 2.990103927 | 0.003298691 | 2.00000003 | 3.03151e-08 | 7.49444e-10 | 6.000000001 | 6.000000272 |

This table uses the notation $\quad \overline{f}_1 = \overline{x}_1 + 3\overline{x}_2 - 3.$

Let us now estimate the degree of influence of the ill-conditioning of the primary problem 4.1.1 on the convergence of the solution process of the system (4.1.1).

To do this, we transform system (4.1.1) into a pair of systems

$$\left\{ \begin{array}{rcl} \overline{\lambda}_i & = & \dfrac{\partial P}{\partial s}\left(\tau, f_i\left(\dfrac{\partial P}{\partial s}\Big(\tau, -g_1\left(\overline{\Lambda}(\tau)\right)\Big), \ldots, \dfrac{\partial P}{\partial s}\Big(\tau, -g_n\left(\overline{\Lambda}(\tau)\right)\Big)\right)\right) \\ \forall i = [1, m]\,, & & \end{array} \right. \tag{4.1.2a}$$

$$\left\{ \begin{array}{rcl} \overline{x}_j & = & \dfrac{\partial P}{\partial s}\left(\tau, -g_j\left(\dfrac{\partial P}{\partial s}\Big(\tau, f_1\left(\overline{x}(\tau)\right)\Big), \ldots, \dfrac{\partial P}{\partial s}\Big(\tau, f_m\left(\overline{x}(\tau)\right)\Big)\right)\right) \\ \forall j = [1, n]\,, & & \end{array} \right. \tag{4.1.2b}$$

which together are equivalent to system (1.2.7) (see (2.2.17) and the derivation for (1.2.7) in §1.2).

Note that this pair of systems can be also written in another, less cumbersome form:

$$\left\{ \begin{array}{rcl} Q\left(\tau, \overline{\lambda}_i\right) & = & f_i\left(\dfrac{\partial P}{\partial s}\Big(\tau, -g_1\left(\overline{\Lambda}(\tau)\right)\Big), \ldots, \dfrac{\partial P}{\partial s}\Big(\tau, -g_n\left(\overline{\Lambda}(\tau)\right)\Big)\right) \\ \forall i = [1, m]\,, & & \end{array} \right. \tag{4.1.3a}$$

$$\left\{ \begin{array}{rcl} Q\left(\tau, \overline{x}_j\right) & = & -g_j\left(\dfrac{\partial P}{\partial s}\Big(\tau, \quad f_1\left(\overline{x}(\tau)\right)\Big), \ldots, \dfrac{\partial P}{\partial s}\Big(\tau, \quad f_m\left(\overline{x}(\tau)\right)\Big)\right) \\ \forall j = [1, n]\,. & & \end{array} \right. \tag{4.1.3b}$$

In systems (4.1.2) and (4.1.3) the first groups of equalities consist of $m$ equations and contain only $m$ unknowns $\overline{\lambda}_1,\ \overline{\lambda}_2,\ \ldots\ ,\ \overline{\lambda}_m$. While the second groups consist of $n$ equations and depend only on $n$ unknowns $\overline{x}_1,\ \overline{x}_2,\ \ldots\ ,\ \overline{x}_n$.

In other words, system (1.2.7) is decomposed into a pair of systems, which can be solved *independently* of each other.

We will solve the primary and dual problems 4.1.1 using systems (4.1.3). Then we compare the convergence of these solutions with the convergence

of the solutions obtained by the feedback function method using system (4.1.1).

For greater clarity, we solve these problems with the same package of algorithms, with the same starting point, but for different values of the parameter $\tau$.

T a b l e 4.1.2

**Comparison of the errors of solving problem 4.1.1 by different versions of the feedback function method**

| $\tau$ | Dual problem (4.1.3a) | primary problem (4.1.3b) | System (4.1.1) |
|---|---|---|---|
| 1.00E-01 | 1.73e-15 | 7.20e-15 | 0 |
| 1.00e-02 | 1.22e-14 | 1.93e-14 | 1.77e-15 |
| 1.00e-03 | 7.27e-14 | 1.30e-12 | 0 |
| 1.00e-04 | 7.29e-14 | 2.14e-12 | 0 |
| 1.00e-05 | 5.17e-10 | 3.16e-04 | 1.62e-05 |
| 1.00e-06 | 0.39 | 1.62e-03 | 5.10e-06 |
| 1.00e-07 | 1.78 | 3.36e-04 | 1.37e-06 |
| 1.00e-08 | 1.85 | 1.37e01 | 4.02e-07 |
| 1.00e-09 | 1.86 | 1.72e04 | 1.25e-07 |
| 1.00e-10 | 4.75e05 | 1.47e05 | 3.93e-08 |
| 1.00e-11 | 1.40 | 6.91e05 | 1.67e-08 |
| 1.00e-12 | 1.20e07 | 6.91e06 | 3.89e-07 |
| 1.00e-13 | 4.35e07 | 3.36e07 | 1.24e-09 |
| 1.00e-14 | 3.47e08 | 1.57e08 | 5.27e-10 |
| 1.00e-15 | 2.23e09 | 3.36e08 | 1.20e-10 |
| 1.00e-16 | 1.08e09 | 1.42e09 | 3.90e-11 |

The columns of Table 4.1.2 show the values of the minimum error that were achieved when solving the pair of problems 4.1.1.

Specifically, these values are the sum of the absolute values of the violations of the equations of the systems being solved at the points for which the solution process was completed (not necessarily successfully!). The gray background marks the variants for which the final value of the error seems acceptable.

The data of Table 4.1.2 show that for $\tau \to +0$ the use of system (4.1.1) in the method of feedback functions allows to obtain better convergence in comparison with systems (4.1.3).

The reason for this difference is that for the variants described in columns 2 and 3 of Table 4.1.2, the primary and dual problems were solved

independently of each other. While for the variant in the 4th column, between the variables of the primary and dual problems, feedback was used.

## 4.2. Modification of feedback functions

### 4.2.1. Alternative types of feedback functions

The conditions that a feedback function must satisfy do not uniquely determine its type. It has been proven that the feedback function method converges for any functions $Q(\tau, s)$ with the properties described in §2.1. However, the choice of a specific type of these functions can affect its efficiency, primarily the rate of convergence to the solution at $\tau \to +0$.

For example, if instead of a feedback function $Q(\tau, s) = \tau\left(s - \frac{1}{s}\right)$ use the function $Q(\tau, s) = \tau\left(s - \frac{1}{s^5}\right)$, then the nature of convergence will change.

This demonstrates the solution of the previously considered example 1.2.1. In it, the stationarity condition of the $U$-function has the form of the system

$$\left\{\begin{array}{rcl} -6 + \overline{x}_1 + 2\overline{x}_2 &=& \tau\left(\overline{\lambda}_1 - \dfrac{1}{\overline{\lambda}_1^5}\right), \\ -6 + 2\overline{x}_1 + \overline{x}_2 &=& \tau\left(\overline{\lambda}_2 - \dfrac{1}{\overline{\lambda}_2^5}\right), \\ -2 + \overline{\lambda}_1 + 2\overline{\lambda}_2 &=& -\tau\left(\overline{x}_1 - \dfrac{1}{\overline{x}_1^5}\right), \\ -3 + 2\overline{\lambda}_1 + \overline{\lambda}_2 &=& -\tau\left(\overline{x}_2 - \dfrac{1}{\overline{x}_2^5}\right), \end{array}\right. \tag{4.2.1}$$

whose solutions for different values of the parameter $\tau$ are given in Table 4.2.1.

Table 4.2.1

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{F}$ | $\overline{G}$ | $\overline{\lambda}_1$ | $\overline{\lambda}_2$ |
|---|---|---|---|---|---|---|
| $10^{-1}$ | 3.480454043 | 1.21112923 | 10.59429578 | 10.86117519 | 1.272542130 | 0.537653735 |
| $10^{-2}$ | 3.250596664 | 1.369250752 | 10.60894559 | 10.08818642 | 1.330250394 | 0.351114009 |
| $10^{-3}$ | 2.160215272 | 1.919343839 | 10.07846206 | 10.00803982 | 1.333874316 | 0.334132320 |
| $10^{-4}$ | 2.016198021 | 1.991846182 | 10.00793459 | 10.00078923 | 1.333397791 | 0.333400410 |
| $10^{-5}$ | 2.001621271 | 1.999183884 | 10.00079420 | 10.00007877 | 1.333339884 | 0.333339910 |
| $10^{-6}$ | 2.000162142 | 1.999918381 | 10.00007943 | 10.00000788 | 1.333333989 | 0.333333990 |
| $10^{-7}$ | 2.000016214 | 1.999991838 | 10.00000794 | 10.00000079 | 1.333333399 | 0.333333399 |
| $10^{-8}$ | 2.000001621 | 1.999999184 | 10.00000079 | 10.00000008 | 1.333333340 | 0.333333340 |
| $10^{-9}$ | 2.000000162 | 1.999999918 | 10.00000008 | 10.00000001 | 1.333333334 | 0.3333333341 |

Comparison of these data with the data in Table 1.2.1a shows that the nature of convergence changes noticeably. In particular, the error in solving the primary problem for the same values of $\tau$ is two orders of magnitude worse.

Therefore, of practical interest may be the construction of alternative types of functions of feedback functions. That is, functions that implement feedback between the variables of the mathematical programming problem and its Lagrange multipliers.

For example, consider the feedback functions $Q(\tau, s)$ defined not by $\forall s > 0$, but by $\forall s \in R$. In this case, we will still assume that the feedback functions on this set:

— are twice continuously differentiable with respect to all their arguments;

— are strictly monotonically increasing and strictly convex with respect to $s$;

— the limit relations are satisfied for them

$$\lim_{s \to -\infty} Q(\tau, s) = -\infty, \qquad \lim_{s \to +\infty} Q(\tau, s) = +\infty .$$

We will demonstrate the fundamental possibility of using such functions in practice.

By direct verification we can verify that, for example,

$$Q_1(\tau, s) = \tau s - \exp\left(-\frac{s}{\tau}\right),$$

$$Q_2(\tau, s) = \left(\tau - \frac{1}{\tau}\right)\sqrt{s^2 + \tau^2} + \left(\tau + \frac{1}{\tau}\right)s \qquad \text{or}$$

$$Q_3(\tau, s) = \ln\left(\exp\left(\tau s\right) + \exp\left(\frac{s}{\tau}\right)\right)$$

have all the necessary properties of feedback functions, intended to take into account the constraint $s \leq 0$.

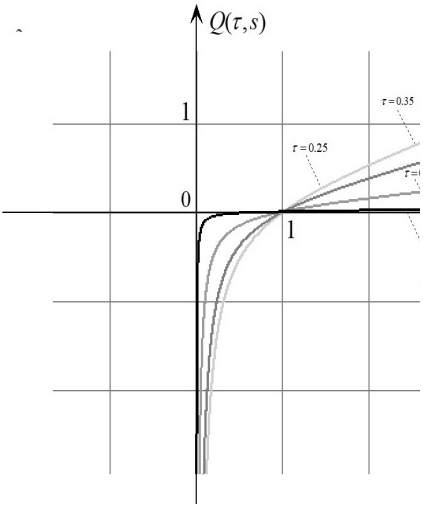



Fig. 4.2.1a. Graphs of functions $\tau(s - 1/s)$

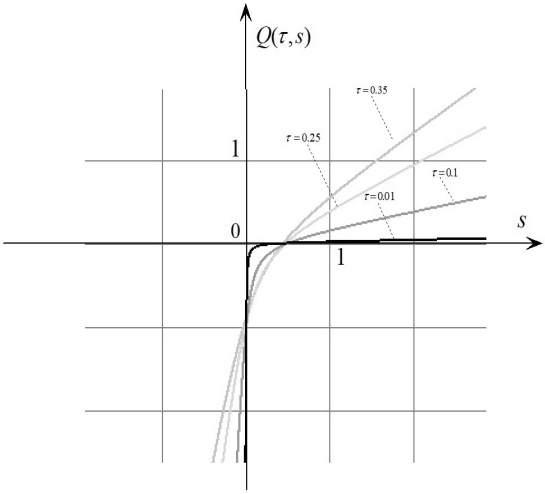



Fig. 4.2.1b. Graphs of functions $Q_2(\tau, s)$

To illustrate the similarities and differences in the properties of the previously used feedback function $Q(\tau, s) = \tau\left(s - \frac{1}{s}\right)$ and the function $Q_2(\tau, s)$, we present their graphs for four different values of the parameter $\tau$ (see Fig. 4.2.1a and Fig. 4.2.1b).

Let us apply the feedback functions of the form $Q_3(\tau, s)$ to solve example 1.2.1 again. Note that in this case $\forall s$ and $\forall \tau > 0$ the inequality will

be true

$$\frac{\partial Q_3}{\partial s} = \frac{\tau \exp(-\tau s) + (1/\tau)\exp(-s/\tau)}{\exp(-\tau s) + \exp(-s/\tau)} > 0\,.$$

T a b l e 4.2.2

| $\tau$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{F}$ | $\overline{G}$ | $\overline{\lambda}_1$ | $\overline{\lambda}_2$ |
|---|---|---|---|---|---|---|
| $10^{-1}$ | 2.031438916 | 1.914618058 | 9.806732006 | 10.78921138 | 1.393259903 | 0.404941993 |
| $10^{-2}$ | 2.002199531 | 1.992200531 | 9.981000656 | 10.07988800 | 1.339940672 | 0.340040662 |
| $10^{-3}$ | 2.000222000 | 1.999222001 | 9.998110001 | 10.00799889 | 1.333999407 | 0.334000407 |
| $10^{-4}$ | 2.000022220 | 1.999922220 | 9.999811100 | 10.00079999 | 1.333399994 | 0.333400004 |
| $10^{-5}$ | 2.000002222 | 1.999992222 | 9.999981111 | 10.00008000 | 1.333340000 | 0.333340000 |
| $10^{-6}$ | 2.000000222 | 1.999999222 | 9.999998111 | 10.00000800 | 1.333334000 | 0.333334000 |
| $10^{-7}$ | 2.000000022 | 1.999999922 | 9.999999811 | 10.00000080 | 1.333333400 | 0.333333400 |
| $10^{-8}$ | 2.000000002 | 1.999999992 | 9.999999981 | 10.00000008 | 1.333333340 | 0.333333340 |
| $10^{-9}$ | 2 | 1.999999999 | 9.999999998 | 10.00000001 | 1.333333334 | 0.333333334 |

The condition of stationarity of the $U$-function in this case has the form of the system

$$\begin{cases} -6 + \overline{x}_1 + 2\overline{x}_2 = \ln\left(\exp\left(\tau\overline{\lambda}_1\right) - \exp\left(-\dfrac{\overline{\lambda}_1}{\tau}\right)\right), \\ -6 + 2\overline{x}_1 + \overline{x}_2 = \ln\left(\exp\left(\tau\overline{\lambda}_2\right) - \exp\left(-\dfrac{\overline{\lambda}_2}{\tau}\right)\right), \\ -2 + \overline{\lambda}_1 + 2\overline{\lambda}_2 = -\ln\left(\exp\left(\tau\overline{x}_1\right) - \exp\left(-\dfrac{\overline{x}_1}{\tau}\right)\right), \\ -3 + 2\overline{\lambda}_1 + \overline{\lambda}_2 = -\ln\left(\exp\left(\tau\overline{x}_2\right) - \exp\left(-\dfrac{\overline{x}_2}{\tau}\right)\right), \end{cases} \tag{4.2.2}$$

whose solutions for various values of the parameter $\tau$ are given in Table 4.2.2.

### 4.2.2. Reducing consumption of computational resources

In some cases, by using special types of feedback functions, it is possible to reduce the overall cost of computational resources.

For example, for feedback functions of the form

$$Q(\tau, s) = \tau\left(s - \frac{1}{s}\right) \qquad \text{or} \qquad Q(\tau, s) = \tau \ln s$$

it is assumed that they are used in the region defined by the condition $s > 0$.

However, in practice, when calculating the values of these functions in the process of solving system (2.1.4), this condition may be violated. The obvious consequence of this will be a violation of the convergence of the computational procedure.

This situation can be avoided if we perform a preliminary check of the inequality $s > 0$ at *trial points*. If it is violated, it is necessary to change the computation algorithm accordingly. The only question is how resource-intensive these changes are.

We will specifically use the fact that the method of feedback functions (according to Theorem 2.1.2) necessarily has a convergence point in the positive orthant of the space $E^m$.

This allows us to replace non-negative unknowns in system (2.1.3) with their absolute values and solve not system (2.1.3), but system

$$\begin{cases} \operatorname{grad}\limits_{x} U\left(\tau, \overline{X}, |\overline{\lambda}_1|, |\overline{\lambda}_2|, \ldots, |\overline{\lambda}_m|\right) = o\,, \\ \operatorname{grad}\limits_{\lambda} U\left(\tau, \overline{X}, |\overline{\lambda}_1|, |\overline{\lambda}_2|, \ldots, |\overline{\lambda}_m|\right) = o\,. \end{cases} \tag{4.2.3}$$

In this case, in the process of solving (4.2.3)) control with respect to the sign of the components $\Lambda$ is not required. True, now the process can end in any orthant $E^m$ . However, the solution of system (2.1.3) will obviously be obtained from the solution of (4.2.3) by simply replacing the found values of the components $\overline{\Lambda}$ with their absolute values.

It is clear that such a modification of the algorithm practically does not lead to an increase in the cost of computing resources. It will also not affect either the rate of convergence or the error of the results. At the same time, it significantly reduces the dependence of the procedure for finding stationary points on the initial approximations.

To illustrate, consider example 1.3.1 again, where the $U$-function has the form

$$U(\tau, X, \Lambda) = \frac{1}{3}x^3 - \frac{4}{3}x - \lambda(-3 + x) + \tau\left(\frac{\lambda^2}{2} - \ln\lambda - \frac{1}{2}\right),$$

and the conditions for its stationarity, that is, system (1.3.2), are, respectively,

$$\begin{cases} \overline{x}^2 - \dfrac{4}{3} - \overline{\lambda} &= 0\,, \\ 3 - \overline{x} &= -\tau\left(\overline{\lambda} - \dfrac{1}{\overline{\lambda}}\right). \end{cases}$$

If we apply the proposed approach to this problem, then it will be necessary to search for stationary points of an auxiliary function of the form

$$\tilde{U}(\tau, X, \Lambda) = \frac{1}{3}x^3 - \frac{4}{3}x - |\lambda|(-3 + x) + \tau\left(\frac{\lambda^2}{2} - \ln|\lambda| - \frac{1}{2}\right),$$

the stationarity conditions for which are determined by the system

$$\begin{cases} \overline{x}^2 - \dfrac{4}{3} - |\overline{\lambda}| &= 0\,, \\ 3 - \overline{x} &= -\tau\left(|\overline{\lambda}| - \dfrac{1}{|\overline{\lambda}|}\right). \end{cases}$$

The domains of definition of the functions $U(\tau, X, \Lambda)$ and $\tilde{U}(\tau, X, \Lambda)$ are different. For $U(\tau, X, \Lambda)$ this domain is defined by the condition $\lambda > 0$, and for $\tilde{U}(\tau, X, \Lambda)$ includes all points, where $\lambda \neq 0$. In this case the values of both functions $\forall\lambda > 0$ are the same.

In Fig. 4.2.2a shows the system of isolines of the function $U(\tau, X, \Lambda)$, and Fig. 4.2.2b for comparison shows the system of isolines for $\tilde{U}(\tau, X, \Lambda)$. Points $A, B$ and $C$ are stationary points, belonging to the *interiors* of the domains of definition of both functions.

For the function $\tilde{U}(\tau, X, \Lambda)$ there are also stationary points $\tilde{A}, \tilde{B}$ and $\tilde{C}$, located symmetrically to the points $A, B$ and $C$ relative to the axis $0x$.

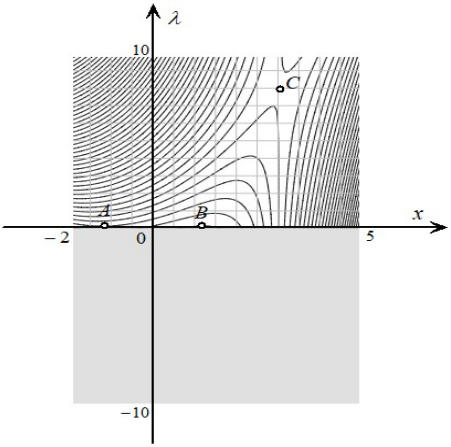



Fig. 4.2.2a. Isoline system functions $U(\tau, X, \Lambda)$

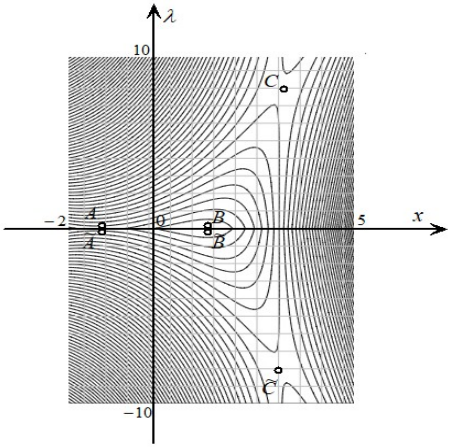



Fig. 4.2.2b. System of isolines functions $\tilde{U}(\tau, X, \Lambda)$

For the pair of points $\{A, \tilde{A}\}$ the symmetry is illustrated by Fig. 4.2.3a. The symmetry of the points $\{B, \tilde{B}\}$ is illustrated by Fig. 4.2.4a.

In Fig. 4.2.3a in the neighborhood of the stationary point $A(X_A^*, \Lambda_A^*)$ the graph of $\tilde{U}(\tau, X_A^*, \Lambda)$ as a function of one variable $\lambda$ is shown.

In Fig. 4.2.3b in the same neighborhood the graph of the function $\tilde{U}(\tau, X, \Lambda_A^*)$ as a function of $x$ is shown. The graphs of Fig. 4.2.3 are given for fixed values of the parameter $\tau = 0.25, \tau = 0.1$ and $\tau = 0.025$.

Fig. 4.2.4a and 4.2.4b present graphs of the same functions for the neighborhood of the stationary point $B(X_B^*, \Lambda_B^*)$.

From Fig. 4.2.3 and 4.2.4 it is clear that the points $\{A, \tilde{A}\}$ and $\{C, \tilde{C}\}$ are stationary points of the *saddle* type. The points $\{B, \tilde{B}\}$ are points of *local minimum*.

## 4.3. Feedback functions in the parametric linearization method

In some cases, the problem can be solved by complicating the formulation, in which an instrumental parameter is introduced.

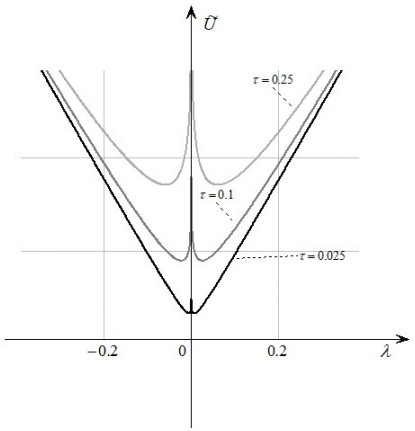
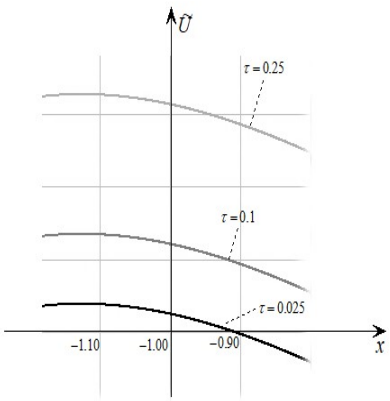



Fig. 4.2.3a. Graphs of functions $\tilde{U}(\tau, X_A^*, \lambda)$




Rice. 4.2.3b. Function graphs $\tilde{U}(\tau, x, \Lambda_A^*)$

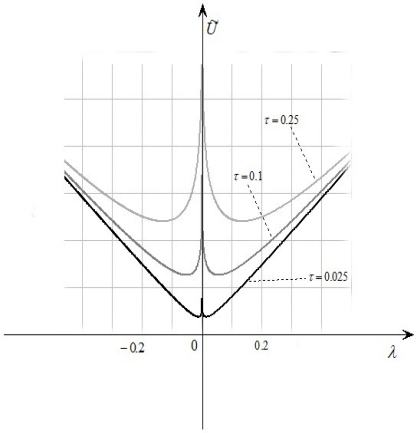



Fig. 4.2.4a. Graphs of functions $\tilde{U}(\tau, X_B^*, \lambda)$

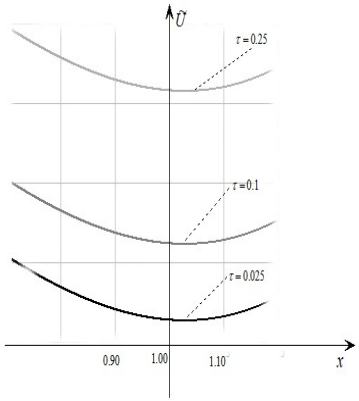



Fig. 4.2.4b. Graphs of functions $\tilde{U}(\tau, x, \Lambda_B^*)$

An example is the method for calculating the Dirichlet integral of the form (0.1.2) described in the introduction.

A possible alternative to adding a new parameter is to turn some of its variables into parameters.

This can be useful if it leads, for example, to a simplification of the

problem formulation or to a decrease in its dimensionality.

An example of the use of this method is the problem of calculating the eigenvalues and eigenvectors of a linear transformation in a finite-dimensional linear space (see (0.1.4)).

As another example of this approach, we consider *parametric linearization*. We shall call this term the case when replacing some variables with parameters turns a nonlinear optimization problem into a linear one.

Let it be required to solve the problem

$$\begin{aligned} &\textit{maximize the function} \qquad F(X) \quad \textit{by } X \in e^{n+K} \\ &\textit{where} \quad \|X\| = \| x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+K} \|^{\mathrm{T}}, \\ &\textit{subject to:} \qquad x_j \geq 0 \; j = \overline{1,n}, \qquad f_i(X) \leq 0 \; i = \overline{1,m}, \end{aligned} \tag{4.3.1}$$

where each of the functions $F(X),\ f_i(X) \quad \forall i = \overline{1,m}$ is twice continuously differentiable with respect to all its arguments.

In addition, let these functions be linear in the variables $x_1, \ldots, x_n$ with the variables $x_{n+1}, \ldots, x_{n+K}$ *fixed*. Let us also assume that $K$ is significantly smaller than $n$. In this case, the variables $x_{n+1}, x_{n+2}, \ldots, x_{n+K}$. can be taken as parameters and the original problem can be considered as a linear parametric one.

For greater clarity, we retain for variables with indices from 1 to $n$ the notation $\|X\| = \| x_1, x_2, \ldots, x_n \|^{\mathrm{T}}$. Let the parameterized variables, which have indices from $n+1$ to $n+K$, be denoted as $\|V\| = \| v_1, v_2, \ldots, v_K \|^{\mathrm{T}}$.

Then the original problem (4.3.1) turns into a two-level system of problems:

high-dimensional lower-level linear problem with a fixed parameter vector $V \in \Omega \subseteq E^K$:

$$\begin{aligned} &\textit{maximize with respect } X \qquad F(X.V) = \sum_{j=1}^{n} \sigma_j(V) x_j \\ &\textit{subject to}: \quad x_j \geq 0 \quad \forall j = \overline{1,n} \\ &f_i(X,V) = -\beta_i(V) + \sum_{j=1}^{n} \alpha_{ij}(V) x_j \leq 0 \qquad i = \overline{1,m} \end{aligned} \tag{4.3.2}$$

and a nonlinear problem (of relatively small dimension) of the upper level in the parameter space $V$:

$$\begin{aligned} &\textit{maximize} \quad F(X_v^*, V) \quad \textit{with respect } V \in e^K \\ &\textit{subject to:} \qquad f_i(X_v^*, V) \leq 0 \qquad \forall i = \overline{1,m}, \end{aligned} \tag{4.3.3}$$

where $X_v^*$ is the solution of problem (4.3.2) for fixed $V \in \Omega$.

For a fixed vector $V$, problem (4.3.2) can be considered a linear programming problem, *dual* to which the problem

$$\begin{aligned} &\textit{minimize by } \Lambda \quad G(\Lambda.V) = \sum_{i=1}^{m} \beta_i(V)\lambda_i \\ &\textit{under conditions}: \quad \lambda_i \geq 0 \quad \forall i = \overline{1,m} \\ &\qquad g_j(\Lambda, V) = -\sigma_j(V) + \sum_{i=1}^{m} \alpha_{ij}(V)\lambda_i \geq 0 \qquad j = \overline{1,n}. \end{aligned} \tag{4.3.4}$$

Accordingly, the solution of problem (4.3.4) for a fixed $V \in \Omega$ will be denoted as $\Lambda_v^*$.

All functions $\sigma_j(V)$, $\beta_i(V)$ and $\alpha_{ij}(V)$ in the conditions of problems (4.3.3)—(4.3.4) are assumed to be known and sufficiently smooth.

Parametric linearization in the case under consideration may be useful, since both for large-scale linear problems and for small-scale nonlinear problems there are effective algorithms for their solution with implementations in the form of computer applications.

To solve problem (4.3.2), we apply the method of feedback functions with an auxiliary $U$-function of the form

$$U(\tau, X, \Lambda, V) = L(X, \Lambda, V) + \sum_{i=1}^{m} R(\tau, \lambda_i) - \sum_{j=1}^{n} R(\tau, x_j)\ ,$$

where $L(X, \Lambda, V)$ is the standard Lagrange function for problem (4.3.2), and $V \in \Omega$.

In expanded form this function will look like

$$\begin{aligned} &U(\tau, X, \Lambda, V) = \\ &= \sum_{j=1}^{n} \Big(\sigma_j(V)x_j - R(\tau, x_j)\Big) + \sum_{i=1}^{m} \Big(\beta_i(V)\lambda_i + R(\tau, \lambda_i)\Big) - \sum_{j=1}^{n}\sum_{i=1}^{m} \alpha_{ij}(V)x_j\lambda_i\ . \end{aligned} \tag{4.3.5}$$

Accordingly, the stationarity conditions of function (4.3.5) with respect to components $X$ and $\Lambda$ are similar to system (1.2.7), namely:

$$\begin{cases} -\beta_i(V) + \sum\limits_{j=1}^{n} \alpha_{ij}(V)\overline{x}_j &= \quad Q(\tau, \overline{\lambda}_i) \qquad i = \overline{1,m}, \\ -\sigma_j(V) + \sum\limits_{i=1}^{m} \alpha_{ij}(V)\overline{\lambda}_i &= -Q(\tau, \overline{x}_i) \qquad j = \overline{1,n}, \end{cases} \tag{4.3.6}$$

where $\dfrac{\partial R}{\partial s}(\tau, s) = Q(\tau, s)$.

Let the vector functions $\overline{X}(\tau, V)$ and $\overline{\Lambda}(\tau, V)$ be the solution of system (4.3.6). We use them as smooth functional approximations of the $X_v^*$ and $\Lambda_v^*$ dependencies to construct an auxiliary function:

$$\overline{U}(\tau, V) = U\Big(\tau,\, \overline{X}(\tau, V),\, \overline{\Lambda}(\tau, V),\, V\Big). \tag{4.3.7}$$

This function, in turn, is a smooth approximation of the objective function of problem (4.3.3). Its stationary points will be approximate approximations of the solutions of this problem in the parameter space.

The search for stationary points of function (4.3.7) by vector $V$ can be performed by any iterative method, using expansions according to Taylor formulas.

As an example, we will apply parametric linearization to the problem of discrete optimal control with mixed-type constraints. This problem was formulated for a mathematical model for assessing the quality of investment management in the securities market.[1]

For the purposes of our course, a particular case of this model, described in [15, 23], will be quite sufficient, where the following operation in the securities market is considered.

During $N$ periods (e.g., banking days), a certain type of securities with a known yield is purchased and sold.

Payment for the purchase of securities is made with borrowed funds attracted through a credit line.

[1]Here it is important to note that this model belongs to the class of so-called *incomplete* mathematical models. That is, mathematical models in which *not all* factors that significantly affect the functioning of the modeled object are correctly taken into account.

The reason for the incompleteness of the model may be both the incomplete volume of reliable information about the object, and the fundamental impossibility of adequately formalizing all significant factors. The latter may include, for example, the human factor, political risks, random events, cause-and-effect uncertainties, etc.

This incompleteness of the model leads to the need to change the formulation of the problems to be solved.

Indeed, any addition to the model, claiming to adequately describe the object, may lead to the optimal solution becoming inadmissible, and the simulation forecast turning out to be incorrect.

Therefore, questions like, «What is the optimal value of the quality criterion if...?» or «What will happen if...?» have to be replaced by questions «What values of the quality criterion are not achievable if...?» or «What will not happen if...?» Since it is obvious that with such replacements, any addition to the model with new conditions cannot transform, for example, an inadmissible solution into an admissible one.

The yield on securities and the cost of credit for each of the periods are constant, but may vary in different periods.

The maximum volumes of funds attracted (or returned) through a credit line in one period, as well as the volumes of securities acquired (or sold) during one period, are limited.

It is assumed that the volume of the securities portfolio and the loan debt are equal to zero both at the beginning of the transaction and at its end.

It is necessary to estimate the maximum efficiency of the transaction by the amount of the transaction account balance at the time of its completion, if the number of periods of the transaction is fixed.

It is also necessary to calculate the dynamics of attracting–returning borrowed funds, as well as the acquisition–sales of securities, ensuring this efficiency. The functional diagram of the operation is shown in Fig. 4.3.1.

For a formalized description of the operation under study, we use the following indicators for each of the periods $k = \overline{0, N}$:

$S(k)$ – current balance of funds on the operation account,
$Q(k)$ – volume of funds attracted (or returned),
$R(k)$ – cost of securities purchased (or sold),
$F(k)$ – volume of securities portfolio,
$V(k)$ – amount of loan debt,
$\rho_0(k)$ – interest rate on loan,
$\rho(k)$ – yield on securities.

These characteristics will be related by the following dynamic relationships for each $k = \overline{0, N-1}$:

1°. Dynamics of the transaction account:

$$S(k+1) = S(k) + Q(k) - R(k) + \rho(k) \cdot F(k) - \rho_0(k) \cdot V(k) \,, \tag{4.3.8}$$

2°. Dynamics of the securities portfolio volume:

$$F(k+1) = F(k) + R(k) \,,$$

3°. Dynamics of the loan debt:

$$V(k+1) = V(k) + Q(k) \,.$$

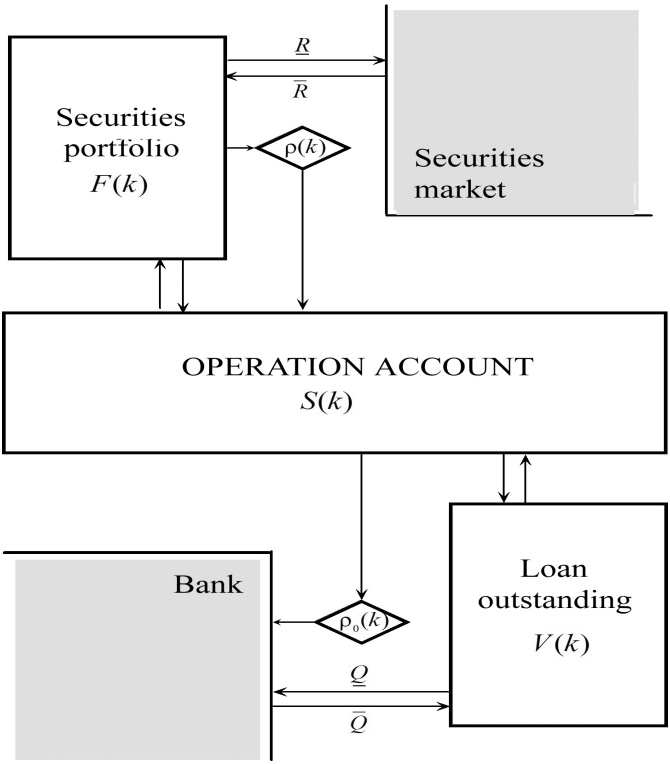



Fig. 4.3.1. Functional diagram of investment operation in the securities market

According to the terminology accepted for dynamic problems, in this system of difference equations, the variables $S$, $F$ and $V$ are *phase variables*, and $\rho$, $\rho_0$, $Q$ and $V$ — *controls*.

The following restrictions are imposed on the phase variables and controls:

— $S(k) \geq 0 \;\; k = \overline{1, N}$, that is, during the operation, an overdraft on the transaction account is not allowed;

— $F(k) \geq 0$ and $V(k) \geq 0 \quad k = \overline{1, N}$, which are obvious;

— controls $R(k)$ and $Q(k)$ are arbitrary in sign and must satisfy bilateral inequalities, reflecting both technical and financial constraints (for example, on the liquidity of securities):

$$\underline{Q}(k) \leq Q(k) \leq \overline{Q}(k) \quad \text{and} \quad \underline{R}(k) \leq R(k) \leq \overline{R}(k) \qquad k = \overline{1, N};$$

— initial conditions: $S(0) = V(0) = F(0) = 0$, as well as final conditions: $V(N) = F(N) = 0$, follow from the conditions of the operation.

Finally, the target condition of the operation is: $S(N) \longrightarrow \max$.

This set of required and desirable conditions is formally a nonlinear programming problem, in which the indicators

$$S(k),\ F(k),\ V(k),\ \rho(k),\ \rho_0(k),\ R(k),\ Q(k) \qquad k = \overline{1, N}$$

are variables, and the number of periods $N$ and the quantities

$$\underline{Q}(k),\ \overline{Q}(k),\ \underline{R}(k),\ \overline{R}(k), \qquad k = \overline{1, N}$$

are parameters, and the quantities $S(k)$, $F(k)$, $V(k)$ have dimension «y. e.», the quantities $R(k)$, $\underline{R}(k)$, $\overline{R}(k)$, $Q(k)$, $\underline{Q}(k)$ and $\overline{Q}(k)$ are measured in «u. e./period», and $\rho(k)$ and $\rho_0(k)$ are dimensionless.

This model obviously does not take into account a number of significant, but non-formalizable factors (for example, such as external economic and political conditions, the human factor, as well as various risks caused by the instability of the securities market or the cost of credit). Therefore, it should be classified as an incomplete model, allowing only a marginal (extreme upper) estimate of the maximum possible efficiency of the operation under consideration.

It is easy to see that *nonlinear* conditions (4.3.8) become linear if the variables $\rho_0(k)$, $\rho(k)$ $k = \overline{1, N}$ are transformed into parameters. That is, to solve the problem, one can use the procedure of two-level parametric linearization.

We will find the solution of the original version of the linearized problem of the lower level for the following set of parameter values: $N = 300$ and

$$\begin{array}{lllllll} \underline{Q}(k) & = & -20.00, & \overline{Q}(k) & = & 40.00, & \\ \underline{R}(k) & = & -50.00, & \overline{R}(k) & = & 40.00, & \\ \rho_0(k) & = & 0.004, & \rho(k) & = & 0.008 & k = \overline{1, N}. \end{array} \tag{4.3.9}$$

The numerical results of calculations obtained by the standard simplex algorithm are presented in Table 5.1.1a–d in the Appendix, § 5.1. This section also provides a brief qualitative analysis of the solution obtained.

Fig. 4.3.2 and Fig. 4.3.3 show the graphs of phase variables and controls, representing the optimal dynamics of their values, ensuring maximization of ! [2] the balance of funds in the transaction account upon its completion.

[2] Assuming the adequacy of the *incomplete* model used.

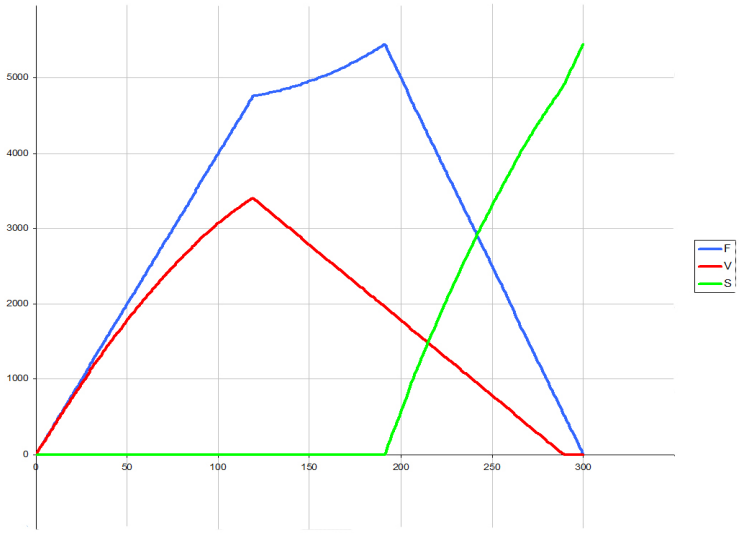



Fig. 4.3.2. Optimal dynamics of *phase* variables in problem (4.3.9)

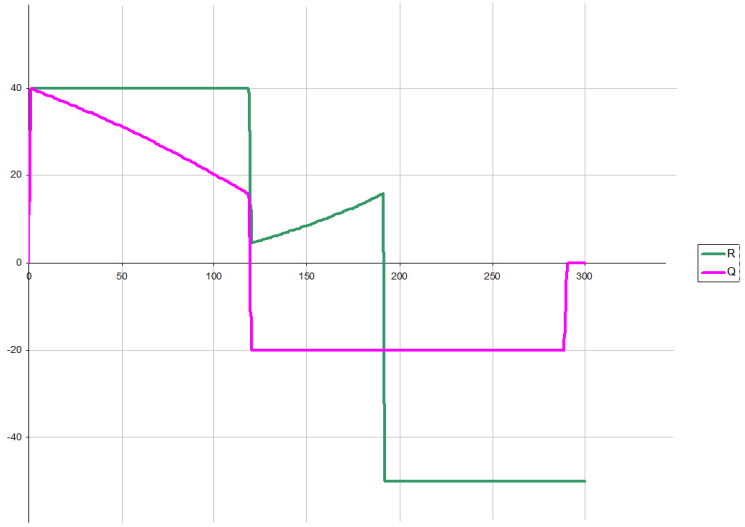



Fig. 4.3.3. Optimal dynamics of control variables in problem 4.3.9

We now consider the case when the yield on securities and the cost of credit are not constant, but are determined by the formulas:

$$\rho(k) = \begin{cases} A\,, & k < K\,, \\ B\,, & K \le k \le K + \Delta\,, \\ A\,, & k > K + \Delta \end{cases}$$

and (4.3.10)

$$\rho_0(k) = \begin{cases} A_0\,, & k < K_0\,, \\ B_0\,, & K_0 \le k \le K_0 + \Delta_0\,, \\ A_0\,, & k > K_0 + \Delta_0\,, \end{cases}$$

that is, the functions $\rho(k)$ and $\rho_0(k)$ are piecewise constant for constant $A$, $B$, $A_0$, $B_0$, $\Delta$ and $\Delta_0$.

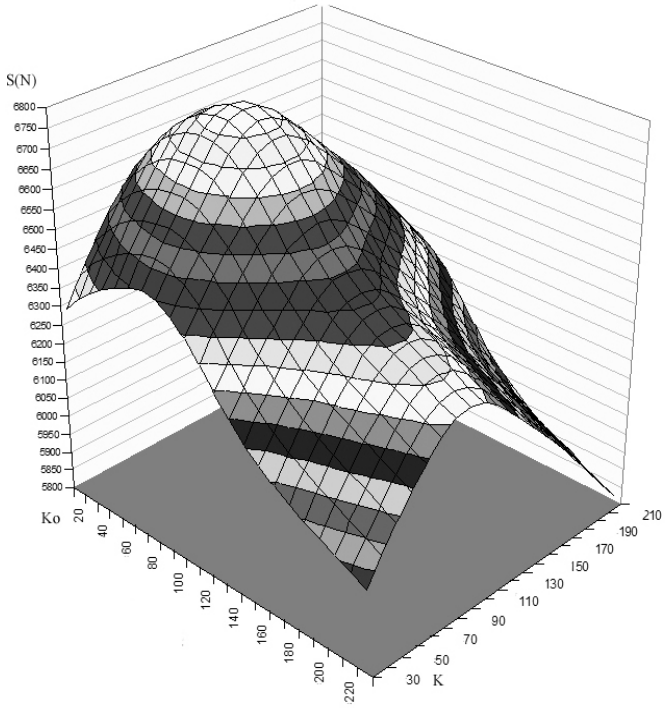



Fig. 4.3.4. 3D representation of the solution to problem (4.3.10)

The problem to be solved consists of analyzing (using the model formulated above) the dependence $S_{K,K_0}(N)$. That is, marginal estimate of the

value of the balance of funds on the transaction account at its completion from the moments of jumps in yield and the cost of credit in the securities market.

Thus, the upper-level problem is nonlinear and is reduced to constructing a description of the dependence of $S(N)$ on $K$ and $K_0$. And since its dimension is small, then for the solution it is possible to use enumeration with respect to the grid of argument values with a constant step.

For the demonstration version, the following parameter values were used:

$$A = 0.008,\ B = 0.01,\ A_0 = 0.004,\ B_0 = 0.006.$$

In this case, the values $N = 300,\quad \Delta = \Delta_0 = 50$ with the parameters of a two-dimensional grid $30 \le K \le 220,\quad 20 \le K_0 \le 230,$ where the grid step was 10.

The calculation results are given in Table 5.2.1a–c, in the Appendix, §5.2. A graphical representation of the dependence $S_{K,K_0}(N)$ is shown in Fig. 4.3.4.

A qualitative analysis of the obtained results shows that in the case of instability of the securities market, manifested in a jump in income, there is a reaction in the credit line market, consisting in a jump in the cost of credit, at which the maximum efficiency of the operation is achieved.

## 4.4. Problems for systems of mathematical models

The scheme of two-level optimization by parameters considered in §3.4 and §3.5 also allows its use for solving problems formulated for a system of mathematical models.

Such problems, for example, arise when decomposing a model of a complex object into a set of models of its subsystems. Or when it is necessary to solve a problem common to a set of models, when there is no possibility of integrating the models into one.

A specific example of a problem of this type can be the problem of optimizing the distribution of production resources between industries or between industry subsystems. In this case, the mathematical model of each subsystem is considered (from the point of view of the managing entity or body) as a "black box", the input parameters of which are the volumes of resources allocated to the subsystem, and the output parameters are the quantitative results of its functioning.

Note that the internal structure, functional diagram, information base, as well as the software used by the subsystem models are independent of each other. Moreover, this data on the subsystems may be unknown both for other subsystems and for the managing entity.

The problem of optimal resource allocation is a two-level parametric problem, at the first (lower) level of which (independently for each of the subsystems) assesses of the efficiency of the subsystem's functioning are calculated for a given vector of allocated resources. While at the second (upper) level, according to these estimates, there are options for redistributing resources between subsystems, allowing to improve the values of the system's performance indicators as a whole.

Such a two-level problem may have a rather non-obvious solution. This is illustrated by the following simple model.

Let's consider a production complex consisting of two subsystems. Each subsystem produces two types of products, consuming two types of resources. Let the first one produce two types of products costing $x_1$ and $x_2$ with resource consumption of $v_1$ and $v_2$. For the second subsystem, these indicators will be equal to $y_1$, $y_2$ and $v_3$, $y_4$ , respectively.

It is known that the production matrices of subsystems are

$$\left\| \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right\| \quad \text{и} \quad \left\| \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right\| .$$

The production capacity for each type of product is limited to a volume of 2, and the cost of the manufactured products for the subsystems is determined by the formulas $4x_1 + 8x_2$ and $4y_1 + 6y_2$ respectively.

Finally, it is required to find the distribution of resources, the total volume of which does not exceed 10, at which the output of products will be maximum.

Thus, a two-level

`Problem 4.4.1` For the system of lower-level problems:

$$\begin{array}{l} \textit{maximize over } X_{(1)} \in E^2 \\ \text{with} \quad \|X_{(1)}\| = \|x_1,\, x_2\|^{\mathrm{T}} \quad \textit{function } F_{(1)} = 4x_1 + 8x_2 \\ \textit{subject to: } 0 \le x_1 \le 2, \quad 0 \le x_2 \le 2, \\ \qquad\qquad x_1 + 2x_2 \le v_1\,, \quad 2x_1 + x_2 \le v_2 \end{array} \tag{4.4.3}$$

and

$$\begin{array}{l} \textit{maximize by } X_{(2)} \in E^2 \\ \text{With} \quad \|X_{(2)}\| = \|y_1,\, y_2\|^{\mathrm{T}} \quad \textit{function } F_{(2)} = 4y_1 + 6y_2 \end{array}$$

$$\textit{subject to:} \quad \begin{array}{l} 0 \le y_1 \le 2, \quad 0 \le y_2 \le 2, \\ 2y_1 + y_2 \le v_1\,, \quad y_1 + 2y_2 \le v_2 \end{array} \tag{4.4.4}$$

solve the following top-level problem:

*find non-negative values* $\{ v_1\,, v_2\,, v_3\,, v_4 \}$, *which maximize the objective function*

$$\Phi_v = 4x^*_{1v} + 8x^*_{2v} + 4y^*_{1v} + 6y^*_{2v} \tag{4.4.5}$$

*subject to:* $v_1 + v_2 + v_3 + v_4 \le 10\,,$ *where the dependencies* $x^*_{1v}, x^*_{2v}, y^*_{1v}$ *and* $y^*_{2v}$ *are the solution of the problems (4.4.3) and (4.4.4).*

This model allows us to study solutions obtained for various hierarchies of objective functions of both individual subsystems and the entire complex as a whole.

Specifically, Table 4.4.1 shows solutions to the problem (4.4.5) for the following variants:

— in the first row: solution variant with the objective function $F_{(1)}$,

— in the second row: with the objective function $F_{(2)}$ ,

— in the third row: with the objective function $\Phi_v$ ,

— finally, in the fourth row: with the objective function of the problem (3.4.4).

We will clarify that in the last case the solution is an equilibrium point on the Pareto set. This point is a compromise for the solutions obtained in the first and second variants.

T a b l e 4.4.1.

| Variables ⇒ Goal ⇓ | $x_1$ | $x_2$ | $y_1$ | $y_2$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $F_{(1)}$ | $F_{(2)}$ | $\Phi_v$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_{(1)} \to \max$ | $\frac{4}{3}$ | 2 | 0 | 0 | $5\frac{1}{3}$ | $4\frac{2}{3}$ | 0 | 0 | $21\frac{1}{3}$ | 0 | $21\frac{1}{3}$ |
| $F_{(2)} \to \max$ | 0 | 0 | $1\frac{1}{3}$ | 2 | 0 | 0 | $4\frac{2}{3}$ | $5\frac{1}{3}$ | 0 | $17\frac{1}{3}$ | $17\frac{1}{3}$ |
| $\Phi_v \to \max$ | 0 | 2 | 0 | $1\frac{1}{3}$ | 4 | 2 | $1\frac{1}{3}$ | $2\frac{2}{3}$ | 16 | 8 | 24 |
| Pareto Equal | 0 | $1\frac{5}{7}$ | 0 | $1\frac{13}{21}$ | $3\frac{3}{7}$ | $1\frac{5}{7}$ | $1\frac{13}{21}$ | $3\frac{5}{21}$ | $13\frac{5}{7}$ | $9\frac{5}{7}$ | $23\frac{3}{7}$ |

The values of the objective functions differ significantly in different variants. Therefore, the problem of optimal choice of resource allocation between subsystems may be of interest in operations research procedures.

Practical construction of a model for combining subsystems is not always possible. As an alternative, one can try to search for the optimal (for the upper level) distribution of resources by analyzing the dependences of solutions of lower-level problems on the parameters.

It is clear that the necessary information would be obtained, for example, from the values of the derivatives of the objective functions of the subsystems by the components of the parameter vector. However, as already noted, the presence of inequality-type constraints in mathematical models makes these dependencies non-differentiable. And it turns out to be necessary to construct algorithms that allow this difficulty to be overcome.

One of the possible ways to solve the problem is to use feedback functions. Let us consider a generalization of the problem statement (4.4.5) for a system of models and describe the proposed approach to its solution.

Let the object under study consist of $N$ subsystems. Each of them is described by a finite-dimensional vector, with components satisfying a system of inequality-type constraints. Let the objective functions of individual subsystems also be known.

In this case, the problem statement for the subsystem with number $s = \overline{1, N}$ is a problem of the form:

*maximize with respect to* $X_{(s)} \in e^{n_{(s)}}$ *the dependence*

$$F_{(s)}\big(X_{(s)},\, V\big) \tag{4.4.6}$$

*under the conditions:* $\qquad f_{(s)i}\big(X_{(s)},\, V\big) \le 0 \qquad i = \overline{1, m}_{(s)}\,.$

Each of them is solved in $E^{n_{(s)}}$ independently of other problems. The parameter vector $V \in E^K$ is fixed and has a coordinate representation $\|V\| = \|\, v_1,\, v_2,\, \ldots,\, v_K \,\|^{\mathrm{T}}$. The functions $F_{(s)}$ and $f_{(s)i}$ are assumed to be twice continuously differentiable with respect to all their arguments.

In addition, let the twice continuously differentiable common objective function $\Phi$ and functions of common (for all subsystems) constraints $\phi_t \quad t = \overline{1, M}$ be given. The function $\Phi$ defines the preference relationship between different states of the system as a whole.

Finally, let the problem, common to the complex of subsystems, be as follows:

*maximize with respect to* $V \in E^K$ *the dependence*

$$\Phi_v = \Phi\Big(X^*_{v(1)}, X^*_{v(2)}, \ldots, X^*_{v(K)}, V\Big) \tag{4.4.7}$$

*under the conditions:* $\varphi_{(v)t} = \varphi_t\Big(X^*_{v(1)}, X^*_{v(2)}, \ldots, X^*_{v(K)}, V\Big) \le 0$
$t = \overline{1, M}$,

where the dependencies $X^*_{v(s)} \quad s = \overline{1, N}$ are the solution to problems (4.4.6).

As before, the problems (4.4.6) are called the problems of *the first* (*lower*) *level*, and the problem (4.4.7) is called the problem of *the second* (*upper*) *level*.

Let us consider the procedure for finding a solution to the upper-level problem, based on the use of feedback functions.

First, we construct an auxiliary $U$-function for each of the models of subsystems (4.4.6):

$$U_{(s)}(\tau, X_{(s)}, \Lambda_{(s)}, V) = F_{(s)}(X_{(s)}, V) - \sum_{i=1}^{m_{(s)}} \lambda_i f_{(s)i}(X_{(s)}, V) + \sum_{i=1}^{m_{(s)}} R(\tau, \lambda_{(s)i}) \tag{4.4.8}$$

or

$$U_{(s)}\big(\tau, X_{(s)}, \Lambda_{(s)}, V\big) = L_{(s)}\big(x_{(s)}, \Lambda_{(s)}, V\big) \; + \; \sum_{i=1}^{m_{(s)}} R(\tau, \lambda_{(s)i}) \; ,$$

where, as before, $\dfrac{\partial R}{\partial s}\,(\tau, s) = Q(\tau, s)$, and $L_{(s)}\big(X_{(s)}, \Lambda_{(s)}, V\big)$ — is the regular Lagrange function. The vector $V$ in these formulas is common to all $s$.

Let us denote by $\overline{X}_{(s)}$ and $\overline{\Lambda}_{(s)}$ the stationary points of the auxiliary functions $U_{(s)}$ with fixed $V$. Under the assumptions made, these stationarity conditions can be written as

$$\begin{cases} \dfrac{\partial U_{(s)}}{\partial x_{(s)j}} & = & 0 & j = \overline{1, n}_{(s)} \; , \\ \dfrac{\partial U_{(s)}}{\partial \lambda_{(s)i}} & = & 0 & i = \overline{1, m}_{(s)} \, , \end{cases} \tag{4.4.9}$$

The system of equations (4.4.9) can be viewed as an implicit definition of the vector functions $\overline{X}_{(s)}(V)$ and $\overline{\Lambda}_{(s)}(V)$.

By virtue of the theorem on a system of functions given implicitly, in some neighborhood of the points $\overline{X}_{(s)}$ and $\overline{\Lambda}_{(s)}$ there will exist continuous derivatives of the form

$$\frac{\partial \overline{x}_{(s)j}}{\partial v_p} \quad j = \overline{1, n}_{(s)} \qquad \text{and} \qquad \frac{\partial \overline{\lambda}_{(s)i}}{\partial v_p} \quad i = \overline{1, m}_{(s)} \quad p = \overline{1, K}\,, \quad (4.4.10)$$

whose values are found, if necessary, $\forall p = \overline{1, K}$ from the following systems of linear equations:

$$\begin{cases} \sum\limits_{t=1}^{n_{(s)}} \dfrac{\partial^2 U_{(s)}}{\partial x_{(s)j} \partial x_{(s)t}} \dfrac{\partial \overline{x}_{(s)t}}{\partial v_p} + \sum\limits_{q=1}^{m_{(s)}} \dfrac{\partial^2 U_{(s)}}{\partial x_{(s)j} \partial \lambda_{(s)q}} \dfrac{\partial \overline{\lambda}_{(s)q}}{\partial v_p} = -\dfrac{\partial^2 U_{(s)}}{\partial x_{(s)j} \partial v_p} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = \overline{1, n}_{(s)}\,, \\ \sum\limits_{t=1}^{n_{(s)}} \dfrac{\partial^2 U_{(s)}}{\partial \lambda_{(s)i} \partial x_{(s)t}} \dfrac{\partial \overline{x}_{(s)t}}{\partial v_p} + \sum\limits_{q=1}^{m_{(s)}} \dfrac{\partial^2 U_{(s)}}{\partial \lambda_{(s)i} \partial \lambda_{(s)q}} \dfrac{\partial \overline{\lambda}_{(s)q}}{\partial v_p} = -\dfrac{\partial^2 U_{(s)}}{\partial \lambda_{(s)i} \partial v_p} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = \overline{1, m}_{(s)}\,. \end{cases} \quad (4.4.11)$$

We now proceed to the study of the upper-level problem (4.4.7).

Let us consider a special case, when the objective function in its statement (4.4.7) is a function of the form

$$\Phi = \sum_{s=1}^{K} F_{(s)}\big(X^*_{(s)v}, V\big),$$

and the system of constraints is obtained by combining the conditions

$$f_{(s)i}\big(X^*_{(s)v},\, V\big) \le 0 \quad i = \overline{1, m}_{(s)}, \qquad s = \overline{1, N}$$

with conditions of the form $Y_t(V) \le 0 \quad t = \overline{1, M}\,.$

Then it is easy to see that, due to the separability of the function $\Phi$, the union of the subsystem models gives a model for which the $U$-function is equal to the sum of functions (4.4.8). In addition, this $U$-function will contain terms that take into account the conditions $Y_t(V) \le 0 \quad t = \overline{1, M}\,.$

To solve problem (4.4.7), we apply the method of feedback functions. That is, we replace $\forall s = \overline{1, N}$ dependencies $X^*_{(s)}(V)$ with their smoothed approximations $\overline{X}_{(s)}(\tau, V)$ and, accordingly, dependencies $\Lambda^*_{(s)}(V)$ with approximations $\overline{\Lambda}_{(s)}(\tau, V)$.

As a result, we will obtain a $\overline{U}$-function for which the stationary point will be an approximate solution of the considered variant of the problem (4.4.7),

$$\overline{U}(\tau,V)=\sum_{s=1}^{N}U_{(s)}\left(\overline{X}_{(s)}(\tau,V),\overline{\Lambda}_{(s)}(\tau,V),V\right)+\sum_{t=1}^{K}\left(\kappa_t Y_t(V)+R(\tau,\kappa_t)\right). \tag{4.4.12}$$

To find these stationary points, we apply the gradient search method in the parameter space $V$. So, at each test point we need to find

$1^\circ$ the value of $\overline{U}(\tau,V)$ and

$2^\circ$ values of partial derivatives $\overline{U}'_{v_r}\ \ r=\overline{1,K}$.

The values of the objective function $\overline{U}(\tau,V)$ are sought using formulas (4.4.12). To calculate its gradient by the components of the parameter vector $V$, we use the rule for differentiating a complex function. As a result, we get the expression:

$$\overline{U}'_{v_r}=\sum_{s=1}^{N}\left(\frac{\partial U_{(s)}}{\partial v_r}+\sum_{j=1}^{n_{(s)}}\frac{\partial U_{(s)}}{\partial x_j}\frac{\partial \overline{x}_j}{\partial v_r}+\sum_{i=1}^{m_{(s)}}\frac{\partial U_{(s)}}{\partial \lambda_i}\frac{\partial \overline{\lambda}_i}{\partial v_r}\right)\quad r=\overline{1,K}. \tag{4.4.13}$$

Finally, it should be noted that these formulas can be simplified by using the stationarity conditions (4.4.9), obtaining as a result

$$\overline{U}'_{v_r}=\sum_{s=1}^{N}\frac{\partial U_{(s)}}{\partial v_r}\qquad r=\overline{1,K}. \tag{4.4.14}$$

We will now demonstrate the practical use of the described method using the example of solving a system of problems, similar to the complex (4.4.3) – (4.4.5).

For greater clarity of the results, we will reduce the dimensionality of the parameter space to two by adding connections between the parameters:

$$v_2=16-v_1,\qquad v_4=15-v_3\,,$$

where $v_1$ and $v_3$ satisfy the constraints

$$0\le v_1\le 16\quad\text{and}\quad 0\le v_3\le 15\,.$$

Then we get a two-level

`Problem 4.4.2` For the system of lower-level problems:

$$\begin{aligned}&\textit{maximize over } X_{(1)} \in E^2\\ &\text{with} \quad \|X_{(1)}\| = \|x_1,\, x_2\|^{\mathrm{T}} \quad \textit{function } F_{(1)} = 4x_1 + 4x_2\\ &\textit{under conditions:} \quad 0 \le x_1 \le 4, \quad 0 \le x_2 \le 2,\\ &\qquad\qquad x_1 + 2x_2 \le v_1\,, \quad 2x_1 + x_2 \le 16 - v_1\end{aligned} \tag{4.4.15}$$

and

$$\begin{aligned}&\textit{maximize by } X_{(2)} \in E^2\\ &\text{c} \quad \|X_{(2)}\| = \|y_1,\, y_2\|^{\mathrm{T}} \quad \textit{function } F_{(2)} = 4y_1 + 3y_2\\ &\textit{subject to:} \quad 0 \le y_1 \le 4, \quad 0 \le y_2 \le 2,\\ &\qquad\qquad 2y_1 + y_2 \le v_3\,, \quad y_1 + 2y_2 \le 15 - v_3\end{aligned} \tag{4.4.16}$$

solve the following top-level problem:

*find non-negative values* $\{\,u\,,\,v\,\}$*, which maximize the objective function*

$$\Phi_v = 4x^*_{1v} + 4x^*_{2v} + 4y^*_{1v} + 3y^*_{2v} \tag{4.4.17}$$

*subject to:* $0 \le u \le 16$ *and* $0 \le v \le 15\,,$ *where the dependencies* $x^*_{1v}, x^*_{2v}, y^*_{1v}$ *and* $y^*_{2v}$ *are the solution to problems (4.4.15) and (4.4.16).*

Table 4.4.2 shows solutions to the problem (4.4.17) for the following variants:

— in the first row: solution variant with the objective function $F_{(1)}$,

— in the second row: with the objective function $F_{(2)}$ ,

— in the third row: with the objective function $\Phi_v$ ,

— finally, in the fourth row: with the objective function of the problem (3.4.4).

In the last case, the solution is an equilibrium point on the Pareto set.

T a b l e 4.4.2.

| Variables ⇒ Goal ⇓ | $x_1$ | $x_2$ | $y_1$ | $y_2$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $F_{(1)}$ | $F_{(2)}$ | $\Phi_v$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_{(1)} \to \max$ | 4 | $1\frac{1}{3}$ | 0 | 0 | $6\frac{2}{3}$ | $9\frac{1}{3}$ | 0 | 0 | $21\frac{12}{3}$ | 0 | $21\frac{1}{3}$ |
| $F_{(2)} \to \max$ | 0 | 0 | 4 | 1 | 0 | 0 | 9 | 6 | 0 | 19 | 19 |
| $\Phi_v \to \max$ | 4 | $1\frac{1}{3}$ | 4 | 1 | $6\frac{2}{3}$ | $9\frac{1}{3}$ | 9 | 6 | $21\frac{1}{3}$ | 19 | $40\frac{1}{3}$ |
| Equal. Pareto | 4 | $1\frac{1}{3}$ | 4 | 1 | $6\frac{2}{3}$ | $9\frac{1}{3}$ | 9 | 6 | $21\frac{1}{3}$ | 19 | $40\frac{1}{3}$ |

Then, according to (4.4.8), the $U$-function for the first subsystem will have the form

$$\begin{aligned} U_{(1)}(\tau,u) = 4x_1+4x_2-\lambda_1\,(x_1+2x_2-v_1)-\lambda_2\,(2x_1+x_2+v_1-16)- \\ -\lambda_3\,(x_1-4)-\lambda_4\,(x_2-2)-R(\tau,x_1)-R(\tau,x_2)+ \\ +R(\tau,\lambda_1)+R(\tau,\lambda_2)+R(\tau,\lambda_3)+R(\tau,\lambda_4)\,. \end{aligned}$$

Here we will use the following specific type of feedback function:

$$Q(\tau,s)=\frac{\partial R}{\partial s}=\frac{\tau}{2}\left(s-\frac{1}{s}\right) \qquad \text{and} \qquad R(\tau,s)=\frac{\tau}{2}\left(\frac{s^2}{2}-\ln s-\frac{1}{2}\right).$$

In this case, the stationarity conditions (4.4.9) $U$-function of the first subsystem for a fixed value of the parameter $v_1$ will be

$$\left\{\begin{array}{rcrcrcl} \dfrac{\partial U_1}{\partial x_1} &=& 4-\overline{\lambda}_1-2\overline{\lambda}_2-\overline{\lambda}_3-\dfrac{\tau}{2}\left(\overline{x}_1-\dfrac{1}{\overline{x}_1}\right) &=& 0\,, \\ \dfrac{\partial U_1}{\partial x_2} &=& 4-2\overline{\lambda}_1-\overline{\lambda}_2-\overline{\lambda}_4-\dfrac{\tau}{2}\left(\overline{x}_2-\dfrac{1}{\overline{x}_2}\right) &=& 0\,, \\ \dfrac{\partial U_1}{\partial \lambda_1} &=& -\overline{x}_1-2\overline{x}_2+v_1+\dfrac{\tau}{2}\left(\overline{\lambda}_1-\dfrac{1}{\overline{\lambda}_1}\right) &=& 0\,, \\ \dfrac{\partial U_1}{\partial \lambda_2} &=& 16-\overline{x}_1-2\overline{x}_2-v_1+\dfrac{\tau}{2}\left(\overline{\lambda}_2-\dfrac{1}{\overline{\lambda}_2}\right) &=& 0\,, \\ \dfrac{\partial U_1}{\partial \lambda_3} &=& 4-\overline{x}_1+\dfrac{\tau}{2}\left(\overline{\lambda}_3-\dfrac{1}{\overline{\lambda}_3}\right) &=& 0\,, \\ \dfrac{\partial U_1}{\partial \lambda_4} &=& 2-\overline{x}_2+\dfrac{\tau}{2}\left(\overline{\lambda}_4-\dfrac{1}{\overline{\lambda}_4}\right) &=& 0\,. \end{array}\right. \tag{4.4.18}$$

Similarly, the $U$-function for the second subsystem will have the form

$$\begin{aligned} U_{(2)}(\tau,u) = 4y_1+3y_2-\mu_1\,(2y_1+y_2-v_3)-\mu_2\,(y_1+2y_2+v_3-15)- \\ -\mu_3\,(y_1-4)-\mu_4\,(y_2-2)-R(\tau,y_1)-R(\tau,y_2)+ \\ +R(\tau,\mu_1)+R(\tau,\mu_2)+R(\tau,\mu_3)+R(\tau,\mu_4)\,. \end{aligned}$$

The stationarity conditions (4.4.7) of the $U$-function of the second subsystem for a fixed value of the parameter $v_3$ will be

$$\left\{\begin{array}{rcrcl}
\dfrac{\partial U_2}{\partial y_1} &=& 4-2\overline{\mu}_1-\overline{\mu}_2-\overline{\mu}_3-\dfrac{\tau}{2}\left(\overline{y}_1-\dfrac{1}{\overline{y}_1}\right) &=& 0\,,\\
\dfrac{\partial U_2}{\partial y_2} &=& 3-\overline{\mu}_1-2\overline{\mu}_2-\overline{\mu}_4-\dfrac{\tau}{2}\left(\overline{y}_2-\dfrac{1}{\overline{y}_2}\right) &=& 0\,,\\
\dfrac{\partial U_2}{\partial \mu_1} &=& -2\overline{y}_1-\overline{y}_2+v_3+\dfrac{\tau}{2}\left(\overline{\mu}_1-\dfrac{1}{\overline{\mu}_1}\right) &=& 0\,,\\
\dfrac{\partial U_2}{\partial \mu_2} &=& 15-\overline{y}_1-2\overline{y}_2-v_3+\dfrac{\tau}{2}\left(\overline{\mu}_2-\dfrac{1}{\overline{\mu}_2}\right) &=& 0\,,\\
\dfrac{\partial U_2}{\partial \mu_3} &=& 4-\overline{y}_1+\dfrac{\tau}{2}\left(\overline{\mu}_3-\dfrac{1}{\overline{\mu}_3}\right) &=& 0\,,\\
\dfrac{\partial U_2}{\partial \mu_4} &=& 2-\overline{y}_2+\dfrac{\tau}{2}\left(\overline{\mu}_4-\dfrac{1}{\overline{\mu}_4}\right) &=& 0\,.
\end{array}\right. \tag{4.4.19}$$

Based on formulas (4.4.12) and the equality

$$U(\tau,v_1,v_3)=U_1(\tau,v_1)+U_2(\tau,v_3)\,, \tag{4.4.20}$$

we obtain that for the considered complex of two subsystems

$$U'_{v_1}=\frac{\partial U}{\partial v_1}=\overline{\lambda}_1-\overline{\lambda}_2,\qquad U'_{v_3}=\frac{\partial U}{\partial v_3}=\overline{\mu}_1-\overline{\mu}_2. \tag{4.4.21}$$

In combination, formulas (4.4.15)—(4.4.21) allow us to implement a gradient method for finding a stationary point for an auxiliary $U$-function in the parameter space $\{v_1,v_3\}$.

The results of calculations for the first few iterations with the initial point $v_{1(0)}=3, v_{3(0)}=3$ at $\tau=0.01$ are given in Table 4.4.3a—4.4.3c. The value *step* — step in the direction $w$, was determined by the dichotomy method.

The following notations are used in the tables:

$$\overline{F}_1=4\overline{x}_1+4\overline{x}_2\,,\qquad \overline{F}_2=4\overline{y}_1+3\overline{y}_2\,,\qquad \overline{F}=\overline{F}_1+\overline{F}_2,$$

$$\bigtriangledown_{v_1}=\frac{\partial U}{\partial v_1},\qquad \bigtriangledown_{v_3}=\frac{\partial U}{\partial v_3},\qquad \|\bigtriangledown\|=\sqrt{\left(\frac{\partial U}{\partial v_1}\right)^2+\left(\frac{\partial U}{\partial v_3}\right)^2},$$

$$w_{v_1} = \frac{\bigtriangledown v_1}{\| \bigtriangledown \|}, \qquad w_{v_3} = \frac{\bigtriangledown v_3}{\| \bigtriangledown \|},$$

$$v_1{}^* = v_1 + step \cdot w_{v_1}, \qquad v_3{}^* = v_3 + step \cdot w_{v_3}.$$

T a b l e 4.4.3a
Finding a stationary point of the $U$-function of Problem 4.4.2
by parameters $v_1$ and $v_3$ by the gradient method

| Iter. | $v_1$ | $v_3$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{F}_1$ | $\overline{\lambda}_1$ | $\overline{\lambda}_2$ | $\overline{\lambda}_3$ | $\overline{\lambda}_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 3.00 | 3.00 | 3.02 | 0.00 | 12.07 | 3.98 | 0.00 | 0.00 | 0.00 |
| 1 | 9.70 | 6.37 | 2.15 | 2.01 | 16.63 | 0.00 | 1.99 | 0.00 | 2.00 |
| 2 | 6.69 | 9.37 | 3.98 | 1.36 | 21.34 | 1.40 | 1.19 | 0.21 | 0.00 |
| 3 | 6.76 | 8.86 | 3.91 | 1.43 | 21.34 | 1.35 | 1.29 | 0.05 | 0.00 |
| 4 | 6.76 | 8.93 | 3.90 | 1.43 | 21.34 | 1.35 | 1.29 | 0.05 | 0.00 |

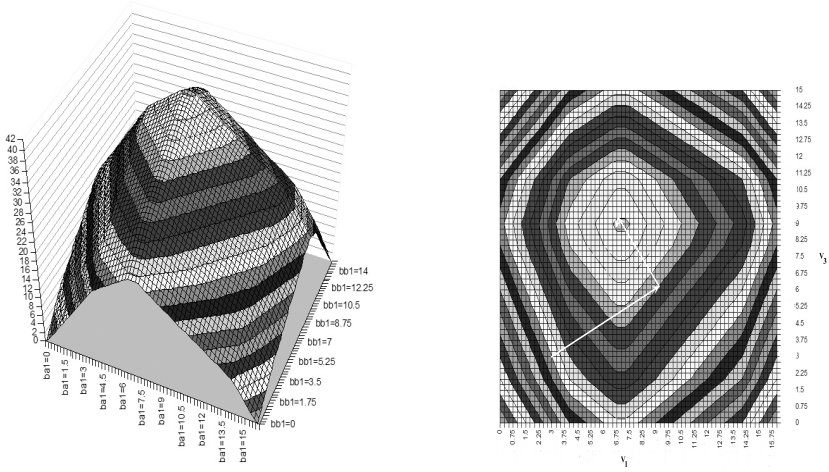
T a b l e 4.4.3b
Search for a stationary point of the $U$-function of Problem 4.4.2
by parameters $v_1$ and $v_3$ by the gradient method

| Iter. | $\overline{y}_1$ | $\overline{y}_2$ | $\overline{F}_2$ | $\overline{\mu}_1$ | $\overline{\mu}_2$ | $\overline{\mu}_3$ | $\overline{\mu}_4$ | $\overline{F}$ | $\overline{U}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.50 | 2.00 | 8.01 | 2.00 | 0.00 | 0.00 | 0.99 | 20.08 | 20.16 |
| 1 | 2.19 | 2.00 | 14.76 | 1.99 | 0.00 | 0.00 | 1.00 | 31.38 | 31.47 |
| 2 | 4.01 | 0.81 | 18.48 | 0.00 | 1.49 | 2.47 | 0.00 | 39.82 | 37.83 |
| 3 | 3.88 | 1.12 | 18.88 | 1.63 | 0.68 | 0.04 | 0.00 | 40.22 | 40.21 |
| 4 | 3.93 | 1.07 | 18.93 | 1.61 | 0.69 | 0.07 | 0.00 | 40.27 | 40.23 |

The graph of the dependence of the objective function of the upper-level problem on the parameters $v_1$ and $v_3$ is shown in Fig. 4.4.1(a).

T a b l e 4.4.3c
Finding a stationary point of the $U$-function of Problem 4.4.2
by parameters $v_1$ and $v_3$ by the gradient method

| Iter. | $\nabla v_1$ | $\nabla v_3$ | $\| \nabla \|$ | $wv_1$ | $wv_3$ | *step* | ${v_1}^*$ | ${v_3}^*$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 3.99 | 2.00 | 4.46 | 0.89 | 0.45 | 7.50 | 9.70 | 6.37 |
| 1 | −1.99 | 1.99 | 2.82 | −0.71 | 0.71 | 4.25 | 6.69 | 9.37 |
| 2 | 0.21 | −1.48 | 1.50 | 0.14 | −0.99 | 0.50 | 6.76 | 8.86 |
| 3 | 0.06 | 0.95 | 0.95 | 0.06 | 1.99 | 0.05 | 6.76 | 8.92 |
| 4 | **** | **** | **** | **** | **** | **** | **** | **** |




(a) (b)

Fig. 4.4.1. Graphical representation of the solution to problem 4.4.2

The trajectory of motion when searching for a stationary point in the parameter space for problem 4.4.2 is shown by white arrows in Fig. 4.4.1(b).

We will now give an example of a case where the objective functions of the upper-level problem differ from the objective functions used in the subsystem models. In this case, we will retain all the previously introduced designations.

**Problem 4.4.3** Find non-negative values of the parameters $\{ v_1 , v_2 , v_3 , v_4 \}$, maximizing the function

$$\Phi_v = 4x^*_{1v} + 8x^*_{2v} + 4y^*_{1v} + 6y^*_{2v}$$

$$F_1 = 4x_1 + 4x_2$$

under conditions: $0 \le x_1 \le 4, \quad 0 \le x_2 \le 2,$
$x_1 + 2x_2 \le v_1 , \quad 2x_1 + x_2 \le v_2 ,$

$$F_2 = 4y_1 + 3y_2$$

when conditions: $0 \le y_1 \le 4, \quad 0 \le y_2 \le 2,$
$2y_1 + y_2 \le v_3 , \quad y_1 + 2y_2 \le v_4 ,$

$0 \le v_1 \le 16, \quad 0 \le v_3 \le 15,$
$v_1 + v_2 = 16, \quad v_3 + v_4 = 15 .$

The solution to problem 4.4.3 for different hierarchies of objective functions is given in Table 4.4.4.

T a b l e 4.4.4.

| Variables ⇒ / Option ⇓ | $x_1$ | $x_2$ | $y_1$ | $y_2$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $F_{(1)}$ | $F_{(2)}$ | $\Phi_v$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_{(1)} \to \max$ | 4 | $1\frac{1}{3}$ | 0 | 0 | $6\frac{2}{3}$ | $9\frac{1}{3}$ | 0 | 0 | $21\frac{12}{3}$ | 0 | $26\frac{2}{3}$ |
| $F_{(2)} \to \max$ | 0 | 0 | 4 | 1 | 0 | 0 | 9 | 6 | 0 | 19 | 22 |
| $\Phi_v \to \max$ | $3\frac{1}{3}$ | 2 | 3 | 2 | $7\frac{1}{3}$ | $8\frac{2}{3}$ | 8 | 7 | $21\frac{1}{3}$ | 18 | $53\frac{1}{3}$ |
| Equal. Pareto | 4 | $1\frac{1}{3}$ | 4 | 1 | $6\frac{2}{3}$ | $9\frac{1}{3}$ | 9 | 6 | $21\frac{1}{3}$ | 19 | $48\frac{1}{3}$ |

As in problem 4.4.2, here we use the standard gradient method using formulas (4.4.13) to find the stationary point of the $U$-function.

It should be noted that for problem 4.4.3, the objective functions in the lower and upper level problems are different. Therefore, the simplified formulas (4.4.14) cannot be used here. We will have to use expressions (4.4.13) directly, calculating in advance at each trial point both the values of the vectors $\overline{x}_{(s)}(\tau, , u)$ and $\overline{\lambda}_{(s)}(\tau, , u)$, and the components of their *sensitivity matrices* $\left\| \dfrac{\partial \overline{x}}{\partial u} \right\|$ and $\left\| \dfrac{\partial \overline{\lambda}}{\partial u} \right\|$. The latter are found from the systems of linear equations (4.4.11).

As before, we will use feedback functions of the form

$$Q(\tau,s)=\frac{\partial R}{\partial s}=\frac{\tau}{2}\left(s-\frac{1}{s}\right)\qquad\text{and}\qquad R(\tau,s)=\frac{\tau}{2}\left(\frac{s^2}{2}-\ln s-\frac{1}{2}\right).$$

The stationarity conditions of the $U$-functions of the subsystems are given by formulas (4.4.18) and (4.4.19). The systems of linear equations (4.4.9) for the first and second lower-level subsystems have the following form, respectively:

$$\left\{\begin{array}{rcl}
-\dfrac{\partial\overline{x}_1}{\partial v_1}-2\dfrac{\partial\overline{x}_2}{\partial v_1}+\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{\lambda}_1^2}\right)\dfrac{\partial\overline{\lambda}_1}{\partial v_1} &=& -1\,,\\
-2\dfrac{\partial\overline{x}_1}{\partial v_1}-\dfrac{\partial\overline{x}_2}{\partial v_1}+\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{\lambda}_2^2}\right)\dfrac{\partial\overline{\lambda}_2}{\partial v_1} &=& 1\,,\\
-\dfrac{\partial\overline{x}_1}{\partial v_1}+\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{\lambda}_3^2}\right)\dfrac{\partial\overline{\lambda}_3}{\partial v_1} &=& 0\,,\\
-\dfrac{\partial\overline{x}_2}{\partial v_1}+\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{\lambda}_4^2}\right)\dfrac{\partial\overline{\lambda}_4}{\partial v_1} &=& 0\,,\\
-\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{x}_1^2}\right)\dfrac{\partial\overline{x}_1}{\partial v_1}-\dfrac{\partial\overline{\lambda}_1}{\partial v_1}-2\dfrac{\partial\overline{\lambda}_2}{\partial v_1}-\dfrac{\partial\overline{\lambda}_3}{\partial v_1} &=& 0\,,\\
-\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{x}_2^2}\right)\dfrac{\partial\overline{x}_2}{\partial v_1}-2\dfrac{\partial\overline{\lambda}_1}{\partial v_1}-\dfrac{\partial\overline{\lambda}_2}{\partial v_1}-\dfrac{\partial\overline{\lambda}_4}{\partial v_1} &=& 0
\end{array}\right.\tag{4.4.22}$$

And

$$\left\{\begin{array}{rcl}
-2\dfrac{\partial\overline{y}_1}{\partial v_3}-\dfrac{\partial\overline{y}_2}{\partial v_3}+\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{\mu}_1^2}\right)\dfrac{\partial\overline{\mu}_1}{\partial v_3} &=& -1\,,\\
-\dfrac{\partial\overline{y}_1}{\partial v_3}-2\dfrac{\partial\overline{y}_2}{\partial v_3}+\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{\mu}_2^2}\right)\dfrac{\partial\overline{\mu}_2}{\partial v_3} &=& 1\,,\\
-\dfrac{\partial\overline{y}_1}{\partial v_3}+\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{\mu}_3^2}\right)\dfrac{\partial\overline{\mu}_3}{\partial v_3} &=& 0\,,\\
-\dfrac{\partial\overline{y}_2}{\partial v_3}+\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{\mu}_4^2}\right)\dfrac{\partial\overline{\mu}_4}{\partial v_3} &=& 0\,,\\
-\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{y}_1^2}\right)\dfrac{\partial\overline{y}_1}{\partial v_3}-2\dfrac{\partial\overline{\mu}_1}{\partial v_3}-\dfrac{\partial\overline{\mu}_2}{\partial v_3}-\dfrac{\partial\overline{\mu}_3}{\partial v_3} &=& 0\,,\\
-\dfrac{\tau}{2}\left(1+\dfrac{1}{\overline{y}_2^2}\right)\dfrac{\partial\overline{y}_2}{\partial v_3}-\dfrac{\partial\overline{\mu}_1}{\partial v_3}-2\dfrac{\partial\overline{\mu}_2}{\partial v_3}-\dfrac{\partial\overline{\mu}_4}{\partial v_3} &=& 0\,.
\end{array}\right.\tag{4.4.23}$$

Using formulas (4.4.13) we obtain that for the complex of two subsys-

tems in problem 4.4.3

$$U'_u = \frac{\partial U}{\partial u} + \frac{\partial U}{\partial x_1}\frac{\partial \overline{x}_1}{\partial u} + \frac{\partial U}{\partial x_2}\frac{\partial \overline{x}_2}{\partial u}, \qquad U'_v = \frac{\partial U}{\partial v} + \frac{\partial U}{\partial x_1}\frac{\partial \overline{x}_1}{\partial v} + \frac{\partial U}{\partial x_2}\frac{\partial \overline{x}_2}{\partial v}, \tag{4.4.24}$$

Where

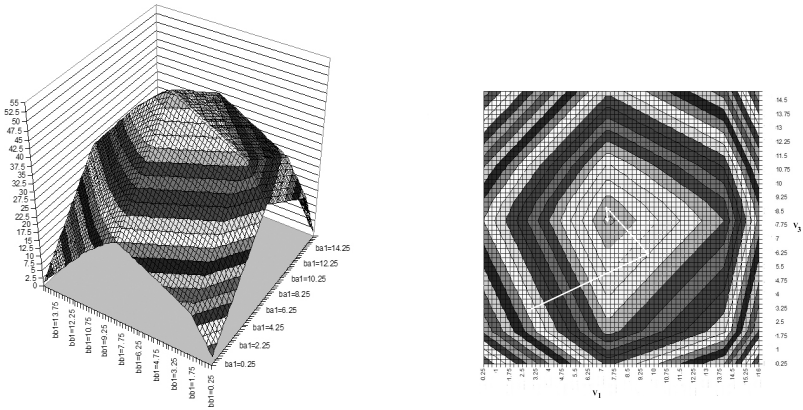
$$\frac{\partial U}{\partial u} = \overline{\lambda}_1 - \overline{\lambda}_2\,, \qquad \frac{\partial U}{\partial v} = \overline{\mu}_1 - \overline{\mu}_2\,,$$

$$\frac{\partial U}{\partial x_1} = 4 - \overline{\lambda}_1 - 2\overline{\lambda}_2 - \frac{\tau}{2}\left(\overline{x}_1 - \frac{1}{\overline{x}_1}\right),$$

$$\frac{\partial U}{\partial x_2} = 8 - 2\overline{\lambda}_1 - \overline{\lambda}_2 - \frac{\tau}{2}\left(\overline{x}_2 - \frac{1}{\overline{x}_2}\right),$$

$$\frac{\partial U}{\partial y_1} = 4 - 2\overline{\mu}_1 - \overline{\mu}_2 - \frac{\tau}{2}\left(\overline{y}_1 - \frac{1}{\overline{y}_1}\right),$$

$$\frac{\partial U}{\partial y_2} = 6 - \overline{\mu}_1 - 2\overline{\mu}_2 - \frac{\tau}{2}\left(\overline{y}_2 - \frac{1}{\overline{y}_2}\right).$$




(a) (b)

Fig. 4.4.2. Graphical representation of the solution of problem 4.4.3

The values of the components of the sensitivity matrices for the iterations performed are contained in Table 4.4.5a—4.4.5b.

The results of calculations for the first few iterations from the initial point $u_0 = 3, v_0 = 3$ at $\tau = 0.01$ are given in Table 4.4.6a—4.4.6c.

For problem 4.4.3 the graphical representation of the dependence of the objective function of the upper-level problem on the parameters $v_1$ and $v_3$ is shown in Fig. 4.4.2(a). The trajectory of motion in the parameter space for this problem is shown in Fig. 4.4.2(b) by white arrows.

T a b l e 4.4.5a.

Sensitivity matrix of the Problem 4.4.3 with respect to the parameter $v_1$

| Iteration | $\frac{\partial \overline{x}_1}{\partial v_1}$ | $\frac{\partial \overline{x}_2}{\partial v_1}$ | $\frac{\partial \overline{\lambda}_1}{\partial v_1}$ | $\frac{\partial \overline{\lambda}_2}{\partial v_1}$ | $\frac{\partial \overline{\lambda}_3}{\partial v_1}$ | $\frac{\partial \overline{\lambda}_4}{\partial v_1}$ |
|---|---|---|---|---|---|---|
| 0 | 1.00 | 0.00 | −0.01 | 0.00 | 0.00 | 0.00 |
| 1 | −0.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 2 | −0.59 | 0.28 | −22.75 | 11.38 | 0.00 | 34.11 |
| 3 | −1.00 | 1.00 | −0.21 | 0.18 | −0.02 | 0.30 |
| 4 | −0.65 | 0.40 | −32.90 | 16.45 | 0.00 | 49.34 |

T a b l e 4.4.5b.

Sensitivity matrix of the Problem 4.4.3 with respect to the parameter $v_3$

| Iteration | $\frac{\partial \overline{y}_1}{\partial v_3}$ | $\frac{\partial \overline{y}_2}{\partial v_3}$ | $\frac{\partial \overline{\mu}_1}{\partial v_3}$ | $\frac{\partial \overline{\mu}_2}{\partial v_3}$ | $\frac{\partial \overline{\mu}_3}{\partial v_3}$ | $\frac{\partial \overline{\mu}_4}{\partial v_3}$ |
|---|---|---|---|---|---|---|
| 0 | 0.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1 | 0.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 2 | 1.00 | −1.00 | −0.05 | 0.04 | 0.07 | 0.00 |
| 3 | 1.00 | −1.00 | −0.11 | 0.21 | 0.00 | −0.31 |
| 4 | 0.99 | −0.99 | −0.47 | 0.93 | 0.00 | −1.38 |

T a b l e 4.4.6a

Finding a stationary point of the $U$-function of Problem 4.4.3

| Iter. | $v_1$ | $v_3$ | $\overline{x}_1$ | $\overline{x}_2$ | $\overline{F}_1$ | $\overline{\lambda}_1$ | $\overline{\lambda}_2$ | $\overline{\lambda}_3$ | $\overline{\lambda}_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 3.00 | 3.00 | 3.02 | 0.00 | 12.07 | 3.98 | 0.00 | 0.00 | 0.00 |
| 1 | 9.70 | 6.37 | 2.15 | 2.01 | 16.64 | 0.00 | 1.99 | 0.00 | 2.00 |
| 2 | 7.35 | 8.72 | 3.33 | 2.00 | 21.32 | 0.34 | 1.82 | 0.00 | −1.49 |
| 3 | 7.20 | 8.12 | 3.46 | 1.87 | 21.32 | 1.31 | 1.33 | 0.01 | 0.04 |
| 4 | 7.34 | 8.05 | 3.33 | 2.00 | 21.33 | 0.49 | 1.74 | 0.01 | −1.27 |

T a b l e 4.4.6b

Finding a stationary point of the $U$-function of the Problem 4.4.3

| Iter. | $\overline{y}_1$ | $\overline{y}_2$ | $\overline{F}_2$ | $\overline{\mu}_1$ | $\overline{\mu}_2$ | $\overline{\mu}_3$ | $\overline{\mu}_4$ | $\overline{F}$ | $\overline{U}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.50 | 2.00 | 8.00 | 2.00 | 0.00 | 0.00 | 0.99 | 26.09 | 26.16 |
| 1 | 2.19 | 2.00 | 14.76 | 1.99 | 0.00 | 0.00 | 1.00 | 31.38 | 31.47 |
| 2 | 3.73 | 1.28 | 18.70 | 1.65 | 0.67 | 0.02 | 0.00 | 51.89 | 51.90 |
| 3 | 3.13 | 1.87 | 18.13 | 1.68 | 0.64 | 0.01 | 0.04 | 52.57 | 52.61 |
| 4 | 3.06 | 1.94 | 18.06 | 1.68 | 0.61 | 0.01 | 0.08 | 53.22 | 53.24 |

T a b l e 4.4.6c

Finding a stationary point of the $U$-function of the Problem 4.4.3

| Iter. | $\nabla v_1$ | $\nabla v_3$ | $\parallel \nabla \parallel$ | $w_{v_1}$ | $w_{v_3}$ | step | ${v_1}^*$ | ${v_3}^*$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 3.98 | 2.00 | 4.45 | 0.89 | 0.45 | 7.50 | 9.70 | 6.37 |
| 1 | −1.99 | 1.99 | 2.82 | −0.71 | 0.71 | 0.62 | 7.20 | 8.12 |
| 2 | −0.49 | −2.02 | 2.08 | −0.23 | −0.97 | 0.50 | 6.76 | 8.72 |
| 3 | 3.97 | −1.97 | 4.43 | 0.90 | −0.44 | 0.16 | 7.34 | 8.05 |
| 4 | 0.35 | −1.89 | 1.92 | **** | **** | **** | **** | **** |

# Chapter 5.

# Appendix

## 5.1. Method of parametric linearization. Solution of the lower-level problem

We will give a detailed description of the solution of problem (4.3.9).

The solution of this version of the lower-level problem (linearized problem) was found for the following set of parameter values: $N = 300$ and

$$\begin{array}{lllllll} \underline{Q}(k) & = & -20.00, & \overline{Q}(k) & = & 40.00, & \\ \underline{R}(k) & = & -50.00, & \overline{R}(k) & = & 40.00, & \\ \rho_0(k) & = & 0.004, & \rho(k) & = & 0.008 & \forall k = \overline{1,N}. \end{array}$$

The numerical results of the calculations obtained by the standard simplex algorithm are given in Table. 5.1.1a—e. Fig. 4.3.2 and Fig. 4.3.3 show graphs of phase variables and controls, representing the optimal dynamics of their values. With this dynamics, the balance of funds on the transaction account at its completion is maximum.

It is convenient to describe the optimal dynamics of phase variables and controls by dividing the set of all periods into four groups.

Periods $1 \le k \le 120$.

For the first group of periods, the volume of the securities portfolio $F$ increases at the maximum possible rate. The value of control $R$ is at the upper limit of its permissible values. The purchase of securities is carried out both at the expense of borrowed funds and reinvestment of dividends from already purchased securities. As the portfolio volume grows, the rate of borrowing $Q$ decreases, although it remains positive. The balance of funds in the account of the operation $S$ for this group of periods is zero.

Periods $121 \le k \le 191$.

In this group of periods, the return of borrowed funds begins. The reduction in the current volume of debt $V$ is carried out at the maximum possible speed. The value of the corresponding control $Q$ is negative and is at its lower limit. At this stage, the growth of the securities portfolio continues, but at a rate lower than the permissible maximum. Dividends are spent primarily on repaying the loan. The rest are reinvested in securities.

Periods from $192 \le k \le 291$.

In this group of periods, the sale of securities occurs. The portfolio volume decreases with the maximum possible speed. This is necessary to satisfy the boundary condition: $F(N) = 0$. The values of both controls are at their lower admissible boundaries. The funds received from the sale of securities and dividends from the remaining part of the portfolio are accumulated in the transaction account. The balance of the transaction account $S$ grows.

Periods $292 \le k \le 300$.

By the beginning of this group of periods, the debt volume becomes equal to zero. Control $Q$ takes a zero value. The growth rate of the balance on the transaction account increases.

It is important to note that the obtained estimate of the maximum efficiency of the operation is *marginal, upper optimistic*. Indeed, adding some model-refining restrictions to the incomplete model cannot expand the set of admissible states. This means that no improvement of the model leads to an increase in the efficiency estimate.

From Fig. 4.3.2—4.3.3 it is clear that the graphs of both phase variables and controls have nonlinear sections. To determine the type of these nonlinearities we use continuous approximations of discrete dependencies by the functions $S(t)$, $F(t)$, $V(t)$, $R(t)$ and $Q(t)$ for values of $t \in [0, T]$.

Let the end of the $k$-th period coincide with the moment $t = k$. Then $T = N$ and the system of difference equations (4.3.8) (1°—3°) can be approximated by a system of differential equations:

$$\begin{cases} \dot{F} = R\,, \\ \dot{V} = Q\,, \\ \dot{S} = \rho F - \rho_0 V - R + Q \end{cases} \tag{5.1.1}$$

under the conditions:

$$F(0) = V(0) = S(0) = 0\,; \qquad F(T) = V(T) = 0\,;$$

$$F(t) \geq 0\,, \quad V(t) \geq 0\,, \quad S(t) \geq 0,$$

$$\underline{R} \leq R(t) \leq \overline{R}\,, \qquad \underline{Q} \leq Q(t) \leq \overline{Q} \qquad \forall t \in [0,T]\,.$$

The objective functional will have the form: $S(T) \longrightarrow \max$.

Consider the first group of periods. Here we have

$$S(t) = 0\,; \;\; R(t) = \overline{R} \quad \Longrightarrow \quad F(t) = \overline{R}t\,.$$

Therefore, system (5.1.1) is simplified and takes the form

$$\begin{cases} \dot{V} = Q\,, \\ \\ 0 = \rho\overline{R}\,t - \rho_0 V - \overline{R} + Q\,, \end{cases}$$

which leads to the *Cauchy problem* for $V(t)$:

$$\dot{V} = \rho_0 V - \rho\overline{R}\,t + \overline{R}, \qquad \mathrm{V(0)}{=}0,$$

Its solution is the function

$$V(t) = \alpha \exp(\rho_0 t) + \beta t + \gamma\,,$$

where $\alpha$, $\beta$ and $\gamma$ are some uniquely determined constants.

Now consider the second group of periods. Assume that the first group ended at time $t_1$. Let $V(t_1) = V_1$ and $F(t_1) = F_1$. Then for the second group we have

$$S(t) = 0\,; \;\; Q(t) = \underline{Q} \qquad \Longrightarrow \qquad V(t) = V_1 + \underline{Q}(t - t_1)\,.$$

Where

$$\begin{cases} \dot{F} = R\,, \\ \\ 0 = \rho F - \rho_0\Big(V_1 + \underline{Q}(t - t_1)\Big) - R + \underline{Q}\,. \end{cases}$$

And we get the Cauchy problem for $F(t)$ :

$$\dot{F} = \rho F - \rho_0\Big(V_1 + \underline{Q}(t - t_1)\Big) + \underline{Q}\,, \qquad F(t_1) = F_1.$$

Its solution will be

$$F(t) = \kappa\, \exp(\rho\, t) + \lambda\, t + \mu \qquad \text{and} \qquad R(t) = \kappa\, \rho\, \exp(\rho\, t) + \lambda\,,$$

where $\kappa$, $\lambda$ and $\mu$ are uniquely determined constants.

In conclusion, let us consider the third[1] group of periods. Let the second group end at time $t_2$ with the values $V(t_2) = V_2$ and $F(t_2) = F_2$. For the third group we have

$$Q(t) = \underline{Q} \qquad \Longrightarrow \qquad V(t) = V_2 + \underline{Q}(t - t_2)\,,$$

$$R(t) = \underline{R} \qquad \Longrightarrow \qquad F(t) = F_2 + \underline{R}(t - t_2)\,.$$

Then system (5.1.1) takes the form

$$\begin{cases} \dot{F} = \underline{R}\,, \\ \dot{V} = \underline{Q}\,, \\ \dot{S} = \rho\Big(F_2 + \underline{R}(t - t_2)\Big) - \rho_0\Big(V_2 + \underline{Q}(t - t_2)\Big) - \underline{R} + \underline{Q}\,. \end{cases}$$

Finally, we obtain the Cauchy problem

$$\dot{S} = \rho\Big(F_2 + \underline{R}(t - t_2)\Big) - \rho_0\Big(V_2 + \underline{Q}(t - t_2)\Big) - \underline{R} + \underline{Q}\,, \qquad S(t_2) = 0\,.$$

Its solution is the function

$$F(t) = \sigma t^2 + \theta t + \nu,$$

where $\sigma, \theta$ and $\nu$ are uniquely determined constants.

Here it is easy to see, for example, that

$$\sigma = \frac{\rho\underline{R} - \rho_0\underline{Q}}{2} \leq 0\,.$$

[1] The study of the fourth group is completely analogous to the study of the third.

T a b l e 5.1.1a

Solution of problem (4.3.9)

| k | $p_o$ | $p_k$ | F | V | S | R | Q |
|---|---|---|---|---|---|---|---|
| 0 | 0.004 | 0.008 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 1 | 0.004 | 0.008 | 40.000 | 39.839 | 0.000 | 40.000 | 39.839 |
| 2 | 0.004 | 0.008 | 80.000 | 79.517 | 0.000 | 40.000 | 39.678 |
| 3 | 0.004 | 0.008 | 120.000 | 119.034 | 0.000 | 40.000 | 39.516 |
| 4 | 0.004 | 0.008 | 160.000 | 158.387 | 0.000 | 40.000 | 39.354 |
| 5 | 0.004 | 0.008 | 200.000 | 197.577 | 0.000 | 40.000 | 39.190 |
| 6 | 0.004 | 0.008 | 240.000 | 236.604 | 0.000 | 40.000 | 39.026 |
| 7 | 0.004 | 0.008 | 280.000 | 275.466 | 0.000 | 40.000 | 38.862 |
| 8 | 0.004 | 0.008 | 320.000 | 314.162 | 0.000 | 40.000 | 38.697 |
| 9 | 0.004 | 0.008 | 360.000 | 352.693 | 0.000 | 40.000 | 38.531 |
| 10 | 0.004 | 0.008 | 400.000 | 391.057 | 0.000 | 40.000 | 38.364 |
| 11 | 0.004 | 0.008 | 440.000 | 429.254 | 0.000 | 40.000 | 38.197 |
| 12 | 0.004 | 0.008 | 480.000 | 467.284 | 0.000 | 40.000 | 38.029 |
| 13 | 0.004 | 0.008 | 520.000 | 505.144 | 0.000 | 40.000 | 37.861 |
| 14 | 0.004 | 0.008 | 560.000 | 542.835 | 0.000 | 40.000 | 37.691 |
| 15 | 0.004 | 0.008 | 600.000 | 580.357 | 0.000 | 40.000 | 37.521 |
| 16 | 0.004 | 0.008 | 640.000 | 617.708 | 0.000 | 40.000 | 37.351 |
| 17 | 0.004 | 0.008 | 680.000 | 654.887 | 0.000 | 40.000 | 37.180 |
| 18 | 0.004 | 0.008 | 720.000 | 691.895 | 0.000 | 40.000 | 37.008 |
| 19 | 0.004 | 0.008 | 760.000 | 728.730 | 0.000 | 40.000 | 36.835 |
| 20 | 0.004 | 0.008 | 800.000 | 765.391 | 0.000 | 40.000 | 36.662 |
| 21 | 0.004 | 0.008 | 840.000 | 801.879 | 0.000 | 40.000 | 36.488 |
| 22 | 0.004 | 0.008 | 880.000 | 838.192 | 0.000 | 40.000 | 36.313 |
| 23 | 0.004 | 0.008 | 920.000 | 874.329 | 0.000 | 40.000 | 36.137 |
| 24 | 0.004 | 0.008 | 960.000 | 910.290 | 0.000 | 40.000 | 35.961 |
| 25 | 0.004 | 0.008 | 1000.000 | 946.074 | 0.000 | 40.000 | 35.784 |
| 26 | 0.004 | 0.008 | 1040.000 | 981.681 | 0.000 | 40.000 | 35.607 |
| 27 | 0.004 | 0.008 | 1080.000 | 1017.110 | 0.000 | 40.000 | 35.428 |
| 28 | 0.004 | 0.008 | 1120.000 | 1052.359 | 0.000 | 40.000 | 35.249 |
| 29 | 0.004 | 0.008 | 1160.000 | 1087.429 | 0.000 | 40.000 | 35.070 |
| 30 | 0.004 | 0.008 | 1200.000 | 1122.318 | 0.000 | 40.000 | 34.889 |
| 31 | 0.004 | 0.008 | 1240.000 | 1157.026 | 0.000 | 40.000 | 34.708 |
| 32 | 0.004 | 0.008 | 1280.000 | 1191.552 | 0.000 | 40.000 | 34.526 |
| 33 | 0.004 | 0.008 | 1320.000 | 1225.896 | 0.000 | 40.000 | 34.344 |
| 34 | 0.004 | 0.008 | 1360.000 | 1260.056 | 0.000 | 40.000 | 34.160 |
| 35 | 0.004 | 0.008 | 1400.000 | 1294.032 | 0.000 | 40.000 | 33.976 |
| 36 | 0.004 | 0.008 | 1440.000 | 1327.823 | 0.000 | 40.000 | 33.791 |
| 37 | 0.004 | 0.008 | 1480.000 | 1361.429 | 0.000 | 40.000 | 33.606 |
| 38 | 0.004 | 0.008 | 1520.000 | 1394.849 | 0.000 | 40.000 | 33.419 |
| 39 | 0.004 | 0.008 | 1560.000 | 1428.081 | 0.000 | 40.000 | 33.232 |
| 40 | 0.004 | 0.008 | 1600.000 | 1461.125 | 0.000 | 40.000 | 33.045 |
| 41 | 0.004 | 0.008 | 1640.000 | 1493.981 | 0.000 | 40.000 | 32.856 |
| 42 | 0.004 | 0.008 | 1680.000 | 1526.648 | 0.000 | 40.000 | 32.667 |
| 43 | 0.004 | 0.008 | 1720.000 | 1559.124 | 0.000 | 40.000 | 32.476 |
| 44 | 0.004 | 0.008 | 1760.000 | 1591.410 | 0.000 | 40.000 | 32.286 |
| 45 | 0.004 | 0.008 | 1800.000 | 1623.504 | 0.000 | 40.000 | 32.094 |
| 46 | 0.004 | 0.008 | 1840.000 | 1655.406 | 0.000 | 40.000 | 31.902 |
| 47 | 0.004 | 0.008 | 1880.000 | 1687.114 | 0.000 | 40.000 | 31.708 |
| 48 | 0.004 | 0.008 | 1920.000 | 1718.629 | 0.000 | 40.000 | 31.515 |
| 49 | 0.004 | 0.008 | 1960.000 | 1749.948 | 0.000 | 40.000 | 31.320 |
| 50 | 0.004 | 0.008 | 2000.000 | 1781.073 | 0.000 | 40.000 | 31.124 |
| 51 | 0.004 | 0.008 | 2040.000 | 1812.001 | 0.000 | 40.000 | 30.928 |
| 52 | 0.004 | 0.008 | 2080.000 | 1842.732 | 0.000 | 40.000 | 30.731 |
| 53 | 0.004 | 0.008 | 2120.000 | 1873.265 | 0.000 | 40.000 | 30.533 |
| 54 | 0.004 | 0.008 | 2160.000 | 1903.599 | 0.000 | 40.000 | 30.334 |
| 55 | 0.004 | 0.008 | 2200.000 | 1933.734 | 0.000 | 40.000 | 30.135 |
| 56 | 0.004 | 0.008 | 2240.000 | 1963.669 | 0.000 | 40.000 | 29.935 |
| 57 | 0.004 | 0.008 | 2280.000 | 1993.402 | 0.000 | 40.000 | 29.734 |
| 58 | 0.004 | 0.008 | 2320.000 | 2022.934 | 0.000 | 40.000 | 29.532 |
| 59 | 0.004 | 0.008 | 2360.000 | 2052.263 | 0.000 | 40.000 | 29.329 |
| 60 | 0.004 | 0.008 | 2400.000 | 2081.389 | 0.000 | 40.000 | 29.126 |
| 61 | 0.004 | 0.008 | 2440.000 | 2110.310 | 0.000 | 40.000 | 28.921 |
| 62 | 0.004 | 0.008 | 2480.000 | 2139.026 | 0.000 | 40.000 | 28.716 |
| 63 | 0.004 | 0.008 | 2520.000 | 2167.536 | 0.000 | 40.000 | 28.510 |
| 64 | 0.004 | 0.008 | 2560.000 | 2195.840 | 0.000 | 40.000 | 28.303 |
| 65 | 0.004 | 0.008 | 2600.000 | 2223.935 | 0.000 | 40.000 | 28.096 |

T a b l e 5.1.1b
Solution of problem (4.3.9)

| k | $p_o$ | $p_k$ | F | V | S | R | Q |
|---|---|---|---|---|---|---|---|
| 66 | 0.004 | 0.008 | 2640.000 | 2251.823 | 0.000 | 40.000 | 27.887 |
| 67 | 0.004 | 0.008 | 2680.000 | 2279.501 | 0.000 | 40.000 | 27.678 |
| 68 | 0.004 | 0.008 | 2720.000 | 2306.968 | 0.000 | 40.000 | 27.468 |
| 69 | 0.004 | 0.008 | 2760.000 | 2334.225 | 0.000 | 40.000 | 27.257 |
| 70 | 0.004 | 0.008 | 2800.000 | 2361.270 | 0.000 | 40.000 | 27.045 |
| 71 | 0.004 | 0.008 | 2840.000 | 2388.103 | 0.000 | 40.000 | 26.832 |
| 72 | 0.004 | 0.008 | 2880.000 | 2414.722 | 0.000 | 40.000 | 26.619 |
| 73 | 0.004 | 0.008 | 2920.000 | 2441.126 | 0.000 | 40.000 | 26.405 |
| 74 | 0.004 | 0.008 | 2960.000 | 2467.316 | 0.000 | 40.000 | 26.189 |
| 75 | 0.004 | 0.008 | 3000.000 | 2493.289 | 0.000 | 40.000 | 25.973 |
| 76 | 0.004 | 0.008 | 3040.000 | 2519.045 | 0.000 | 40.000 | 25.756 |
| 77 | 0.004 | 0.008 | 3080.000 | 2544.583 | 0.000 | 40.000 | 25.538 |
| 78 | 0.004 | 0.008 | 3120.000 | 2569.903 | 0.000 | 40.000 | 25.320 |
| 79 | 0.004 | 0.008 | 3160.000 | 2595.003 | 0.000 | 40.000 | 25.100 |
| 80 | 0.004 | 0.008 | 3200.000 | 2619.882 | 0.000 | 40.000 | 24.880 |
| 81 | 0.004 | 0.008 | 3240.000 | 2644.540 | 0.000 | 40.000 | 24.658 |
| 82 | 0.004 | 0.008 | 3280.000 | 2668.976 | 0.000 | 40.000 | 24.436 |
| 83 | 0.004 | 0.008 | 3320.000 | 2693.189 | 0.000 | 40.000 | 24.213 |
| 84 | 0.004 | 0.008 | 3360.000 | 2717.178 | 0.000 | 40.000 | 23.989 |
| 85 | 0.004 | 0.008 | 3400.000 | 2740.942 | 0.000 | 40.000 | 23.764 |
| 86 | 0.004 | 0.008 | 3440.000 | 2764.480 | 0.000 | 40.000 | 23.538 |
| 87 | 0.004 | 0.008 | 3480.000 | 2787.791 | 0.000 | 40.000 | 23.311 |
| 88 | 0.004 | 0.008 | 3520.000 | 2810.874 | 0.000 | 40.000 | 23.083 |
| 89 | 0.004 | 0.008 | 3560.000 | 2833.729 | 0.000 | 40.000 | 22.855 |
| 90 | 0.004 | 0.008 | 3600.000 | 2856.355 | 0.000 | 40.000 | 22.625 |
| 91 | 0.004 | 0.008 | 3640.000 | 2878.750 | 0.000 | 40.000 | 22.395 |
| 92 | 0.004 | 0.008 | 3680.000 | 2900.913 | 0.000 | 40.000 | 22.164 |
| 93 | 0.004 | 0.008 | 3720.000 | 2922.845 | 0.000 | 40.000 | 21.931 |
| 94 | 0.004 | 0.008 | 3760.000 | 2944.543 | 0.000 | 40.000 | 21.698 |
| 95 | 0.004 | 0.008 | 3800.000 | 2966.007 | 0.000 | 40.000 | 21.464 |
| 96 | 0.004 | 0.008 | 3840.000 | 2987.236 | 0.000 | 40.000 | 21.229 |
| 97 | 0.004 | 0.008 | 3880.000 | 3008.229 | 0.000 | 40.000 | 20.993 |
| 98 | 0.004 | 0.008 | 3920.000 | 3028.985 | 0.000 | 40.000 | 20.756 |
| 99 | 0.004 | 0.008 | 3960.000 | 3049.503 | 0.000 | 40.000 | 20.518 |
| 100 | 0.004 | 0.008 | 4000.000 | 3069.782 | 0.000 | 40.000 | 20.279 |
| 101 | 0.004 | 0.008 | 4040.000 | 3089.821 | 0.000 | 40.000 | 20.039 |
| 102 | 0.004 | 0.008 | 4080.000 | 3109.619 | 0.000 | 40.000 | 19.798 |
| 103 | 0.004 | 0.008 | 4120.000 | 3129.176 | 0.000 | 40.000 | 19.557 |
| 104 | 0.004 | 0.008 | 4160.000 | 3148.490 | 0.000 | 40.000 | 19.314 |
| 105 | 0.004 | 0.008 | 4200.000 | 3167.560 | 0.000 | 40.000 | 19.070 |
| 106 | 0.004 | 0.008 | 4240.000 | 3186.386 | 0.000 | 40.000 | 18.826 |
| 107 | 0.004 | 0.008 | 4280.000 | 3204.966 | 0.000 | 40.000 | 18.580 |
| 108 | 0.004 | 0.008 | 4320.000 | 3223.299 | 0.000 | 40.000 | 18.333 |
| 109 | 0.004 | 0.008 | 4360.000 | 3241.385 | 0.000 | 40.000 | 18.086 |
| 110 | 0.004 | 0.008 | 4400.000 | 3259.221 | 0.000 | 40.000 | 17.837 |
| 111 | 0.004 | 0.008 | 4440.000 | 3276.809 | 0.000 | 40.000 | 17.587 |
| 112 | 0.004 | 0.008 | 4480.000 | 3294.145 | 0.000 | 40.000 | 17.337 |
| 113 | 0.004 | 0.008 | 4520.000 | 3311.230 | 0.000 | 40.000 | 17.085 |
| 114 | 0.004 | 0.008 | 4560.000 | 3328.062 | 0.000 | 40.000 | 16.832 |
| 115 | 0.004 | 0.008 | 4600.000 | 3344.641 | 0.000 | 40.000 | 16.579 |
| 116 | 0.004 | 0.008 | 4640.000 | 3360.965 | 0.000 | 40.000 | 16.324 |
| 117 | 0.004 | 0.008 | 4680.000 | 3377.033 | 0.000 | 40.000 | 16.068 |
| 118 | 0.004 | 0.008 | 4720.000 | 3392.844 | 0.000 | 40.000 | 15.811 |
| 119 | 0.004 | 0.008 | 4757.126 | 3405.536 | 0.000 | 37.126 | 12.691 |
| 120 | 0.004 | 0.008 | 4761.677 | 3385.536 | 0.000 | 4.551 | -20.000 |
| 121 | 0.004 | 0.008 | 4766.346 | 3365.536 | 0.000 | 4.669 | -20.000 |
| 122 | 0.004 | 0.008 | 4771.133 | 3345.536 | 0.000 | 4.787 | -20.000 |
| 123 | 0.004 | 0.008 | 4776.039 | 3325.536 | 0.000 | 4.906 | -20.000 |
| 124 | 0.004 | 0.008 | 4781.066 | 3305.536 | 0.000 | 5.026 | -20.000 |
| 125 | 0.004 | 0.008 | 4786.213 | 3285.536 | 0.000 | 5.148 | -20.000 |
| 126 | 0.004 | 0.008 | 4791.483 | 3265.536 | 0.000 | 5.270 | -20.000 |
| 127 | 0.004 | 0.008 | 4796.876 | 3245.536 | 0.000 | 5.393 | -20.000 |
| 128 | 0.004 | 0.008 | 4802.393 | 3225.536 | 0.000 | 5.517 | -20.000 |
| 129 | 0.004 | 0.008 | 4808.035 | 3205.536 | 0.000 | 5.642 | -20.000 |
| 130 | 0.004 | 0.008 | 4813.803 | 3185.536 | 0.000 | 5.768 | -20.000 |

T a b l e 5.1.1c
Solution of problem (4.3.9)

| k | $p_o$ | $p_k$ | F | V | S | R | Q |
|---|---|---|---|---|---|---|---|
| 131 | 0.004 | 0.008 | 4819.699 | 3165.536 | 0.000 | 5.895 | -20.000 |
| 132 | 0.004 | 0.008 | 4825.722 | 3145.536 | 0.000 | 6.024 | -20.000 |
| 133 | 0.004 | 0.008 | 4831.875 | 3125.536 | 0.000 | 6.153 | -20.000 |
| 134 | 0.004 | 0.008 | 4838.158 | 3105.536 | 0.000 | 6.283 | -20.000 |
| 135 | 0.004 | 0.008 | 4844.573 | 3085.536 | 0.000 | 6.414 | -20.000 |
| 136 | 0.004 | 0.008 | 4851.119 | 3065.536 | 0.000 | 6.547 | -20.000 |
| 137 | 0.004 | 0.008 | 4857.800 | 3045.536 | 0.000 | 6.680 | -20.000 |
| 138 | 0.004 | 0.008 | 4864.614 | 3025.536 | 0.000 | 6.815 | -20.000 |
| 139 | 0.004 | 0.008 | 4871.565 | 3005.536 | 0.000 | 6.950 | -20.000 |
| 140 | 0.004 | 0.008 | 4878.652 | 2985.536 | 0.000 | 7.087 | -20.000 |
| 141 | 0.004 | 0.008 | 4885.877 | 2965.536 | 0.000 | 7.225 | -20.000 |
| 142 | 0.004 | 0.008 | 4893.241 | 2945.536 | 0.000 | 7.364 | -20.000 |
| 143 | 0.004 | 0.008 | 4900.744 | 2925.536 | 0.000 | 7.504 | -20.000 |
| 144 | 0.004 | 0.008 | 4908.389 | 2905.536 | 0.000 | 7.645 | -20.000 |
| 145 | 0.004 | 0.008 | 4916.177 | 2885.536 | 0.000 | 7.787 | -20.000 |
| 146 | 0.004 | 0.008 | 4924.107 | 2865.536 | 0.000 | 7.931 | -20.000 |
| 147 | 0.004 | 0.008 | 4932.183 | 2845.536 | 0.000 | 8.075 | -20.000 |
| 148 | 0.004 | 0.008 | 4940.404 | 2825.536 | 0.000 | 8.221 | -20.000 |
| 149 | 0.004 | 0.008 | 4948.772 | 2805.536 | 0.000 | 8.368 | -20.000 |
| 150 | 0.004 | 0.008 | 4957.288 | 2785.536 | 0.000 | 8.516 | -20.000 |
| 151 | 0.004 | 0.008 | 4965.953 | 2765.536 | 0.000 | 8.665 | -20.000 |
| 152 | 0.004 | 0.008 | 4974.769 | 2745.536 | 0.000 | 8.816 | -20.000 |
| 153 | 0.004 | 0.008 | 4983.737 | 2725.536 | 0.000 | 8.968 | -20.000 |
| 154 | 0.004 | 0.008 | 4992.858 | 2705.536 | 0.000 | 9.121 | -20.000 |
| 155 | 0.004 | 0.008 | 5002.133 | 2685.536 | 0.000 | 9.275 | -20.000 |
| 156 | 0.004 | 0.008 | 5011.563 | 2665.536 | 0.000 | 9.430 | -20.000 |
| 157 | 0.004 | 0.008 | 5021.150 | 2645.536 | 0.000 | 9.587 | -20.000 |
| 158 | 0.004 | 0.008 | 5030.895 | 2625.536 | 0.000 | 9.745 | -20.000 |
| 159 | 0.004 | 0.008 | 5040.800 | 2605.536 | 0.000 | 9.904 | -20.000 |
| 160 | 0.004 | 0.008 | 5050.864 | 2585.536 | 0.000 | 10.065 | -20.000 |
| 161 | 0.004 | 0.008 | 5061.091 | 2565.536 | 0.000 | 10.227 | -20.000 |
| 162 | 0.004 | 0.008 | 5071.481 | 2545.536 | 0.000 | 10.390 | -20.000 |
| 163 | 0.004 | 0.008 | 5082.035 | 2525.536 | 0.000 | 10.554 | -20.000 |
| 164 | 0.004 | 0.008 | 5092.755 | 2505.536 | 0.000 | 10.720 | -20.000 |
| 165 | 0.004 | 0.008 | 5103.642 | 2485.536 | 0.000 | 10.887 | -20.000 |
| 166 | 0.004 | 0.008 | 5114.697 | 2465.536 | 0.000 | 11.055 | -20.000 |
| 167 | 0.004 | 0.008 | 5125.922 | 2445.536 | 0.000 | 11.225 | -20.000 |
| 168 | 0.004 | 0.008 | 5137.319 | 2425.536 | 0.000 | 11.396 | -20.000 |
| 169 | 0.004 | 0.008 | 5148.888 | 2405.536 | 0.000 | 11.569 | -20.000 |
| 170 | 0.004 | 0.008 | 5160.631 | 2385.536 | 0.000 | 11.743 | -20.000 |
| 171 | 0.004 | 0.008 | 5172.549 | 2365.536 | 0.000 | 11.918 | -20.000 |
| 172 | 0.004 | 0.008 | 5184.644 | 2345.536 | 0.000 | 12.095 | -20.000 |
| 173 | 0.004 | 0.008 | 5196.917 | 2325.536 | 0.000 | 12.273 | -20.000 |
| 174 | 0.004 | 0.008 | 5209.370 | 2305.536 | 0.000 | 12.453 | -20.000 |
| 175 | 0.004 | 0.008 | 5222.004 | 2285.536 | 0.000 | 12.634 | -20.000 |
| 176 | 0.004 | 0.008 | 5234.820 | 2265.536 | 0.000 | 12.816 | -20.000 |
| 177 | 0.004 | 0.008 | 5247.821 | 2245.536 | 0.000 | 13.000 | -20.000 |
| 178 | 0.004 | 0.008 | 5261.006 | 2225.536 | 0.000 | 13.186 | -20.000 |
| 179 | 0.004 | 0.008 | 5274.379 | 2205.536 | 0.000 | 13.373 | -20.000 |
| 180 | 0.004 | 0.008 | 5287.941 | 2185.536 | 0.000 | 13.561 | -20.000 |
| 181 | 0.004 | 0.008 | 5301.692 | 2165.536 | 0.000 | 13.751 | -20.000 |
| 182 | 0.004 | 0.008 | 5315.635 | 2145.536 | 0.000 | 13.943 | -20.000 |
| 183 | 0.004 | 0.008 | 5329.771 | 2125.536 | 0.000 | 14.136 | -20.000 |
| 184 | 0.004 | 0.008 | 5344.102 | 2105.536 | 0.000 | 14.331 | -20.000 |
| 185 | 0.004 | 0.008 | 5358.629 | 2085.536 | 0.000 | 14.527 | -20.000 |
| 186 | 0.004 | 0.008 | 5373.353 | 2065.536 | 0.000 | 14.725 | -20.000 |
| 187 | 0.004 | 0.008 | 5388.277 | 2045.536 | 0.000 | 14.924 | -20.000 |
| 188 | 0.004 | 0.008 | 5403.402 | 2025.536 | 0.000 | 15.125 | -20.000 |
| 189 | 0.004 | 0.008 | 5418.730 | 2005.536 | 0.000 | 15.328 | -20.000 |
| 190 | 0.004 | 0.008 | 5434.262 | 1985.536 | 0.000 | 15.532 | -20.000 |
| 191 | 0.004 | 0.008 | 5450.000 | 1965.536 | 0.000 | 15.738 | -20.000 |
| 192 | 0.004 | 0.008 | 5400.000 | 1945.536 | 65.418 | -50.000 | -20.000 |
| 193 | 0.004 | 0.008 | 5350.000 | 1925.536 | 130.516 | -50.000 | -20.000 |
| 194 | 0.004 | 0.008 | 5300.000 | 1905.536 | 195.294 | -50.000 | -20.000 |
| 195 | 0.004 | 0.008 | 5250.000 | 1885.536 | 259.751 | -50.000 | -20.000 |
| 196 | 0.004 | 0.008 | 5200.000 | 1865.536 | 323.889 | -50.000 | -20.000 |
| 197 | 0.004 | 0.008 | 5150.000 | 1845.536 | 387.707 | -50.000 | -20.000 |
| 198 | 0.004 | 0.008 | 5100.000 | 1825.536 | 451.205 | -50.000 | -20.000 |
| 199 | 0.004 | 0.008 | 5050.000 | 1805.536 | 514.383 | -50.000 | -20.000 |
| 200 | 0.004 | 0.008 | 5000.000 | 1785.536 | 577.241 | -50.000 | -20.000 |

T a b l e 5.1.1d
Solution of problem (4.3.9)

| k | $p_o$ | $p_k$ | F | V | S | R | Q |
|---|---|---|---|---|---|---|---|
| 201 | 0.004 | 0.008 | 4950.000 | 1765.536 | 639.779 | -50.000 | -20.000 |
| 202 | 0.004 | 0.008 | 4900.000 | 1745.536 | 701.996 | -50.000 | -20.000 |
| 203 | 0.004 | 0.008 | 4850.000 | 1725.536 | 763.894 | -50.000 | -20.000 |
| 204 | 0.004 | 0.008 | 4800.000 | 1705.536 | 825.472 | -50.000 | -20.000 |
| 205 | 0.004 | 0.008 | 4750.000 | 1685.536 | 886.730 | -50.000 | -20.000 |
| 206 | 0.004 | 0.008 | 4700.000 | 1665.536 | 947.668 | -50.000 | -20.000 |
| 207 | 0.004 | 0.008 | 4650.000 | 1645.536 | 1008.286 | -50.000 | -20.000 |
| 208 | 0.004 | 0.008 | 4600.000 | 1625.536 | 1068.584 | -50.000 | -20.000 |
| 209 | 0.004 | 0.008 | 4550.000 | 1605.536 | 1128.561 | -50.000 | -20.000 |
| 210 | 0.004 | 0.008 | 4500.000 | 1585.536 | 1188.219 | -50.000 | -20.000 |
| 211 | 0.004 | 0.008 | 4450.000 | 1565.536 | 1247.557 | -50.000 | -20.000 |
| 212 | 0.004 | 0.008 | 4400.000 | 1545.536 | 1306.575 | -50.000 | -20.000 |
| 213 | 0.004 | 0.008 | 4350.000 | 1525.536 | 1365.273 | -50.000 | -20.000 |
| 214 | 0.004 | 0.008 | 4300.000 | 1505.536 | 1423.651 | -50.000 | -20.000 |
| 215 | 0.004 | 0.008 | 4250.000 | 1485.536 | 1481.709 | -50.000 | -20.000 |
| 216 | 0.004 | 0.008 | 4200.000 | 1465.536 | 1539.446 | -50.000 | -20.000 |
| 217 | 0.004 | 0.008 | 4150.000 | 1445.536 | 1596.864 | -50.000 | -20.000 |
| 218 | 0.004 | 0.008 | 4100.000 | 1425.536 | 1653.962 | -50.000 | -20.000 |
| 219 | 0.004 | 0.008 | 4050.000 | 1405.536 | 1710.740 | -50.000 | -20.000 |
| 220 | 0.004 | 0.008 | 4000.000 | 1385.536 | 1767.198 | -50.000 | -20.000 |
| 221 | 0.004 | 0.008 | 3950.000 | 1365.536 | 1823.336 | -50.000 | -20.000 |
| 222 | 0.004 | 0.008 | 3900.000 | 1345.536 | 1879.154 | -50.000 | -20.000 |
| 223 | 0.004 | 0.008 | 3850.000 | 1325.536 | 1934.651 | -50.000 | -20.000 |
| 224 | 0.004 | 0.008 | 3800.000 | 1305.536 | 1989.829 | -50.000 | -20.000 |
| 225 | 0.004 | 0.008 | 3750.000 | 1285.536 | 2044.687 | -50.000 | -20.000 |
| 226 | 0.004 | 0.008 | 3700.000 | 1265.536 | 2099.225 | -50.000 | -20.000 |
| 227 | 0.004 | 0.008 | 3650.000 | 1245.536 | 2153.443 | -50.000 | -20.000 |
| 228 | 0.004 | 0.008 | 3600.000 | 1225.536 | 2207.341 | -50.000 | -20.000 |
| 229 | 0.004 | 0.008 | 3550.000 | 1205.536 | 2260.919 | -50.000 | -20.000 |
| 230 | 0.004 | 0.008 | 3500.000 | 1185.536 | 2314.176 | -50.000 | -20.000 |
| 231 | 0.004 | 0.008 | 3450.000 | 1165.536 | 2367.114 | -50.000 | -20.000 |
| 232 | 0.004 | 0.008 | 3400.000 | 1145.536 | 2419.732 | -50.000 | -20.000 |
| 233 | 0.004 | 0.008 | 3350.000 | 1125.536 | 2472.030 | -50.000 | -20.000 |
| 234 | 0.004 | 0.008 | 3300.000 | 1105.536 | 2524.008 | -50.000 | -20.000 |
| 235 | 0.004 | 0.008 | 3250.000 | 1085.536 | 2575.666 | -50.000 | -20.000 |
| 236 | 0.004 | 0.008 | 3200.000 | 1065.536 | 2627.004 | -50.000 | -20.000 |
| 237 | 0.004 | 0.008 | 3150.000 | 1045.536 | 2678.021 | -50.000 | -20.000 |
| 238 | 0.004 | 0.008 | 3100.000 | 1025.536 | 2728.719 | -50.000 | -20.000 |
| 239 | 0.004 | 0.008 | 3050.000 | 1005.536 | 2779.097 | -50.000 | -20.000 |
| 240 | 0.004 | 0.008 | 3000.000 | 985.536 | 2829.155 | -50.000 | -20.000 |
| 241 | 0.004 | 0.008 | 2950.000 | 965.536 | 2878.893 | -50.000 | -20.000 |
| 242 | 0.004 | 0.008 | 2900.000 | 945.536 | 2928.311 | -50.000 | -20.000 |
| 243 | 0.004 | 0.008 | 2850.000 | 925.536 | 2977.409 | -50.000 | -20.000 |
| 244 | 0.004 | 0.008 | 2800.000 | 905.536 | 3026.186 | -50.000 | -20.000 |
| 245 | 0.004 | 0.008 | 2750.000 | 885.536 | 3074.644 | -50.000 | -20.000 |
| 246 | 0.004 | 0.008 | 2700.000 | 865.536 | 3122.782 | -50.000 | -20.000 |
| 247 | 0.004 | 0.008 | 2650.000 | 845.536 | 3170.600 | -50.000 | -20.000 |
| 248 | 0.004 | 0.008 | 2600.000 | 825.536 | 3218.098 | -50.000 | -20.000 |
| 249 | 0.004 | 0.008 | 2550.000 | 805.536 | 3265.276 | -50.000 | -20.000 |
| 250 | 0.004 | 0.008 | 2500.000 | 785.536 | 3312.134 | -50.000 | -20.000 |
| 251 | 0.004 | 0.008 | 2450.000 | 765.536 | 3358.671 | -50.000 | -20.000 |
| 252 | 0.004 | 0.008 | 2400.000 | 745.536 | 3404.889 | -50.000 | -20.000 |
| 253 | 0.004 | 0.008 | 2350.000 | 725.536 | 3450.787 | -50.000 | -20.000 |
| 254 | 0.004 | 0.008 | 2300.000 | 705.536 | 3496.365 | -50.000 | -20.000 |
| 255 | 0.004 | 0.008 | 2250.000 | 685.536 | 3541.623 | -50.000 | -20.000 |
| 256 | 0.004 | 0.008 | 2200.000 | 665.536 | 3586.561 | -50.000 | -20.000 |
| 257 | 0.004 | 0.008 | 2150.000 | 645.536 | 3631.179 | -50.000 | -20.000 |
| 258 | 0.004 | 0.008 | 2100.000 | 625.536 | 3675.476 | -50.000 | -20.000 |
| 259 | 0.004 | 0.008 | 2050.000 | 605.536 | 3719.454 | -50.000 | -20.000 |
| 260 | 0.004 | 0.008 | 2000.000 | 585.536 | 3763.112 | -50.000 | -20.000 |
| 261 | 0.004 | 0.008 | 1950.000 | 565.536 | 3806.450 | -50.000 | -20.000 |
| 262 | 0.004 | 0.008 | 1900.000 | 545.536 | 3849.468 | -50.000 | -20.000 |
| 263 | 0.004 | 0.008 | 1850.000 | 525.536 | 3892.166 | -50.000 | -20.000 |
| 264 | 0.004 | 0.008 | 1800.000 | 505.536 | 3934.544 | -50.000 | -20.000 |
| 265 | 0.004 | 0.008 | 1750.000 | 485.536 | 3976.601 | -50.000 | -20.000 |
| 266 | 0.004 | 0.008 | 1700.000 | 465.536 | 4018.339 | -50.000 | -20.000 |
| 267 | 0.004 | 0.008 | 1650.000 | 445.536 | 4059.757 | -50.000 | -20.000 |
| 268 | 0.004 | 0.008 | 1600.000 | 425.536 | 4100.855 | -50.000 | -20.000 |
| 269 | 0.004 | 0.008 | 1550.000 | 405.536 | 4141.633 | -50.000 | -20.000 |

T a b l e 5.1.1e
Solution of problem (4.3.9)

| k | $p_o$ | $p_k$ | F | V | S | R | Q |
|---|---|---|---|---|---|---|---|
| 270 | 0.004 | 0.008 | 1500.000 | 385.536 | 4182.091 | -50.000 | -20.000 |
| 271 | 0.004 | 0.008 | 1450.000 | 365.536 | 4222.229 | -50.000 | -20.000 |
| 272 | 0.004 | 0.008 | 1400.000 | 345.536 | 4262.046 | -50.000 | -20.000 |
| 273 | 0.004 | 0.008 | 1350.000 | 325.536 | 4301.544 | -50.000 | -20.000 |
| 274 | 0.004 | 0.008 | 1300.000 | 305.536 | 4340.722 | -50.000 | -20.000 |
| 275 | 0.004 | 0.008 | 1250.000 | 285.536 | 4379.580 | -50.000 | -20.000 |
| 276 | 0.004 | 0.008 | 1200.000 | 265.536 | 4418.118 | -50.000 | -20.000 |
| 277 | 0.004 | 0.008 | 1150.000 | 245.536 | 4456.336 | -50.000 | -20.000 |
| 278 | 0.004 | 0.008 | 1100.000 | 225.536 | 4494.234 | -50.000 | -20.000 |
| 279 | 0.004 | 0.008 | 1050.000 | 205.536 | 4531.811 | -50.000 | -20.000 |
| 280 | 0.004 | 0.008 | 1000.000 | 185.536 | 4569.069 | -50.000 | -20.000 |
| 281 | 0.004 | 0.008 | 950.000 | 165.536 | 4606.007 | -50.000 | -20.000 |
| 282 | 0.004 | 0.008 | 900.000 | 145.536 | 4642.625 | -50.000 | -20.000 |
| 283 | 0.004 | 0.008 | 850.000 | 125.536 | 4678.923 | -50.000 | -20.000 |
| 284 | 0.004 | 0.008 | 800.000 | 105.536 | 4714.901 | -50.000 | -20.000 |
| 285 | 0.004 | 0.008 | 750.000 | 85.536 | 4750.559 | -50.000 | -20.000 |
| 286 | 0.004 | 0.008 | 700.000 | 65.536 | 4785.896 | -50.000 | -20.000 |
| 287 | 0.004 | 0.008 | 650.000 | 45.536 | 4820.914 | -50.000 | -20.000 |
| 288 | 0.004 | 0.008 | 600.000 | 25.536 | 4855.612 | -50.000 | -20.000 |
| 289 | 0.004 | 0.008 | 550.000 | 5.536 | 4889.990 | -50.000 | -20.000 |
| 290 | 0.004 | 0.008 | 500.000 | 0.000 | 4938.454 | -50.000 | -5.536 |
| 291 | 0.004 | 0.008 | 450.000 | 0.000 | 4992.054 | -50.000 | 0.000 |
| 292 | 0.004 | 0.008 | 400.000 | 0.000 | 5045.254 | -50.000 | 0.000 |
| 293 | 0.004 | 0.008 | 350.000 | 0.000 | 5098.054 | -50.000 | 0.000 |
| 294 | 0.004 | 0.008 | 300.000 | 0.000 | 5150.454 | -50.000 | 0.000 |
| 295 | 0.004 | 0.008 | 250.000 | 0.000 | 5202.454 | -50.000 | 0.000 |
| 296 | 0.004 | 0.008 | 200.000 | 0.000 | 5254.054 | -50.000 | 0.000 |
| 297 | 0.004 | 0.008 | 150.000 | 0.000 | 5305.254 | -50.000 | 0.000 |
| 298 | 0.004 | 0.008 | 100.000 | 0.000 | 5356.054 | -50.000 | 0.000 |
| 299 | 0.004 | 0.008 | 50.000 | 0.000 | 5406.454 | -50.000 | 0.000 |
| 300 | 0.004 | 0.008 | 0.000 | 0.000 | 5456.454 | -50.000 | 0.000 |

## 5.2. Method of parametric linearization. Solution of the upper-level problem

The solution of the upper-level problem (4.3.10) consists of finding the dependence of the balance of funds $S(N)$ as a function of $K_0$ and $K$.

Here we consider the case when the yield on securities and the cost of credit are not constant, but are determined by the formulas:

$$\rho(k) = \begin{cases} A\,, & k < K\,, \\ B\,, & K \le k \le K + \Delta\,, \\ A\,, & k > K + \Delta \end{cases}$$

and

$$\rho_0(k) = \begin{cases} A_0\,, & k < K_0\,, \\ B_0\,, & K_0 \le k \le K_0 + \Delta_0\,, \\ A_0\,, & k > K_0 + \Delta_0\,. \end{cases}$$

In the example considered above, the upper-level problem, due to its small dimension, was solved by the enumeration method with respect to a coordinate grid with a fixed step.

The following parameter values were used:

$$A = 0.008, \quad B = 0.010,$$

$$A_0 = 0.004, \; B_0 = 0.006.$$

The value of the jump length $\Delta = \Delta_0 = 50$. The grid parameters were chosen $30 \le K \le 220, \quad 20 \le K_0 \le 230,$ with a grid step size of 10.

From the obtained solution, it follows that in the case of unstable behavior of the securities market, that is, with a jump in yield, an optimal response of the credit line market is possible. In other words, it is possible to choose the moment of the start of the jump in the cost of credit, which will ensure maximum investment efficiency.

T a b l e 5.2.1a

| $\downarrow K, K_o \rightarrow$ | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
|---|---|---|---|---|---|---|---|
| 30 | 6301.553 | 6345.351 | 6379.959 | 6407.948 | 6430.167 | 6447.357 | 6456.531 |
| 40 | 6373.044 | 6417.783 | 6451.449 | 6476.777 | 6496.386 | 6511.013 | 6518.022 |
| 50 | 6437.814 | 6483.969 | 6518.951 | 6543.269 | 6560.000 | 6571.849 | 6576.506 |
| 60 | 6495.743 | 6543.449 | 6580.005 | 6605.996 | 6621.879 | 6630.813 | 6632.842 |
| 70 | 6546.458 | 6596.085 | 6634.667 | 6662.604 | 6680.367 | 6688.449 | 6687.650 |
| 80 | 6586.890 | 6638.279 | 6679.245 | 6709.470 | 6729.502 | 6739.777 | 6738.712 |
| 90 | 6607.775 | 6659.498 | 6702.454 | 6735.502 | 6758.163 | 6770.951 | 6772.066 |
| 100 | 6609.973 | 6662.159 | 6705.445 | 6740.526 | 6766.187 | 6781.803 | 6785.357 |
| 110 | 6594.758 | 6647.533 | 6691.336 | 6726.852 | 6754.659 | 6773.367 | 6779.597 |
| 120 | 6563.670 | 6617.072 | 6661.330 | 6697.224 | 6725.381 | 6746.342 | 6755.412 |
| 130 | 6522.109 | 6575.706 | 6620.231 | 6656.400 | 6684.762 | 6705.903 | 6716.318 |
| 140 | 6480.797 | 6534.481 | 6579.082 | 6615.269 | 6643.678 | 6664.861 | 6675.015 |
| 150 | 6439.331 | 6493.038 | 6537.699 | 6573.926 | 6602.354 | 6623.553 | 6633.315 |
| 160 | 6396.835 | 6450.572 | 6495.252 | 6531.505 | 6559.902 | 6581.059 | 6590.467 |
| 170 | 6352.549 | 6406.335 | 6450.995 | 6487.174 | 6515.513 | 6536.633 | 6545.592 |
| 180 | 6305.836 | 6359.572 | 6404.093 | 6440.177 | 6468.507 | 6489.638 | 6498.140 |
| 190 | 6256.017 | 6309.540 | 6354.033 | 6390.153 | 6418.449 | 6439.534 | 6447.656 |
| 200 | 6205.017 | 6258.540 | 6303.033 | 6339.153 | 6367.449 | 6388.534 | 6396.656 |
| 210 | 6154.017 | 6207.540 | 6252.033 | 6288.153 | 6316.449 | 6337.534 | 6345.656 |
| 220 | 6103.017 | 6156.540 | 6201.033 | 6237.153 | 6265.449 | 6286.534 | 6294.656 |

Table 5.2.1b

| $\downarrow K, K_o \rightarrow$ | 90 | 100 | 110 | 120 | 130 | 140 | 150 |
|---|---|---|---|---|---|---|---|
| 30 | 6449.670 | 6427.582 | 6392.573 | 6346.765 | 6293.637 | 6245.539 | 6204.915 |
| 40 | 6510.274 | 6488.833 | 6454.588 | 6409.579 | 6357.115 | 6309.504 | 6269.311 |
| 50 | 6566.627 | 6544.278 | 6510.752 | 6466.549 | 6414.733 | 6367.547 | 6327.743 |
| 60 | 6620.595 | 6596.034 | 6561.532 | 6518.053 | 6466.816 | 6420.028 | 6380.501 |
| 70 | 6672.902 | 6645.836 | 6608.923 | 6564.396 | 6513.700 | 6467.275 | 6428.077 |
| 80 | 6721.630 | 6692.250 | 6653.002 | 6606.126 | 6554.322 | 6508.331 | 6469.520 |
| 90 | 6754.852 | 6723.276 | 6681.763 | 6632.558 | 6578.640 | 6532.036 | 6493.736 |
| 100 | 6770.021 | 6738.063 | 6694.114 | 6642.272 | 6586.132 | 6538.161 | 6499.285 |
| 110 | 6766.361 | 6736.067 | 6691.321 | 6636.629 | 6578.285 | 6528.830 | 6488.743 |
| 120 | 6744.442 | 6715.908 | 6672.597 | 6616.940 | 6556.074 | 6504.933 | 6463.501 |
| 130 | 6707.069 | 6679.893 | 6637.642 | 6582.915 | 6521.222 | 6468.477 | 6425.689 |
| 140 | 6666.341 | 6640.380 | 6599.174 | 6545.343 | 6484.094 | 6430.697 | 6386.361 |
| 150 | 6624.262 | 6598.818 | 6558.629 | 6505.662 | 6445.167 | 6392.022 | 6347.147 |
| 160 | 6580.962 | 6555.160 | 6515.415 | 6463.252 | 6403.758 | 6350.957 | 6306.290 |
| 170 | 6535.671 | 6509.508 | 6469.495 | 6417.644 | 6359.192 | 6306.745 | 6262.469 |
| 180 | 6487.788 | 6461.196 | 6420.839 | 6368.850 | 6310.953 | 6259.110 | 6215.282 |
| 190 | 6437.085 | 6410.383 | 6369.894 | 6317.818 | 6260.049 | 6208.375 | 6164.643 |
| 200 | 6386.085 | 6359.383 | 6318.894 | 6266.818 | 6209.049 | 6157.375 | 6113.643 |
| 210 | 6335.085 | 6308.383 | 6267.894 | 6215.818 | 6158.049 | 6106.375 | 6062.643 |
| 220 | 6284.085 | 6257.383 | 6216.894 | 6164.818 | 6107.049 | 6055.375 | 6011.643 |

Table 5.2.1c

| $K, K_o \rightarrow$ | 160 | 170 | 180 | 190 | 200 | 210 | 220 | 230 |
|---|---|---|---|---|---|---|---|---|
| 30 | 6170.748 | 6142.021 | 6117.668 | 6096.756 | 6076.356 | 6055.956 | 6035.556 | 6015.156 |
| 40 | 6235.508 | 6206.961 | 6182.882 | 6161.976 | 6141.576 | 6121.176 | 6100.776 | 6080.376 |
| 50 | 6294.170 | 6265.878 | 6241.998 | 6221.154 | 6200.754 | 6180.354 | 6159.954 | 6139.554 |
| 60 | 6347.210 | 6319.174 | 6295.331 | 6274.602 | 6254.202 | 6233.802 | 6213.402 | 6193.002 |
| 70 | 6395.041 | 6367.194 | 6343.498 | 6322.772 | 6302.372 | 6281.972 | 6261.572 | 6241.172 |
| 80 | 6436.799 | 6409.298 | 6385.853 | 6365.257 | 6344.857 | 6324.457 | 6304.057 | 6283.657 |
| 90 | 6461.518 | 6434.433 | 6411.624 | 6391.186 | 6370.786 | 6350.386 | 6329.986 | 6309.586 |
| 100 | 6467.601 | 6441.024 | 6418.633 | 6398.233 | 6377.833 | 6357.433 | 6337.033 | 6316.633 |
| 110 | 6456.503 | 6430.330 | 6408.499 | 6388.099 | 6367.699 | 6347.299 | 6326.899 | 6306.499 |
| 120 | 6430.251 | 6403.778 | 6382.230 | 6361.830 | 6341.430 | 6321.030 | 6300.630 | 6280.230 |
| 130 | 6391.288 | 6363.795 | 6341.920 | 6321.520 | 6301.120 | 6280.720 | 6260.320 | 6239.920 |
| 140 | 6350.585 | 6321.876 | 6299.101 | 6278.701 | 6258.301 | 6237.901 | 6217.501 | 6197.101 |
| 150 | 6309.984 | 6280.197 | 6256.378 | 6235.930 | 6215.530 | 6195.130 | 6174.730 | 6154.330 |
| 160 | 6268.698 | 6237.919 | 6213.050 | 6192.363 | 6171.963 | 6151.563 | 6131.163 | 6110.763 |
| 170 | 6225.257 | 6194.043 | 6168.347 | 6147.230 | 6126.830 | 6106.430 | 6086.030 | 6065.630 |
| 180 | 6178.372 | 6147.326 | 6121.456 | 6099.837 | 6079.437 | 6059.037 | 6038.637 | 6018.237 |
| 190 | 6127.915 | 6097.077 | 6071.263 | 6049.506 | 6029.106 | 6008.706 | 5988.306 | 5967.906 |
| 200 | 6076.915 | 6046.077 | 6020.263 | 5998.506 | 5978.106 | 5957.706 | 5937.306 | 5916.906 |
| 210 | 6025.915 | 5995.077 | 5969.263 | 5947.506 | 5927.106 | 5906.706 | 5886.306 | 5865.906 |
| 220 | 5974.915 | 5944.077 | 5918.263 | 5896.506 | 5876.106 | 5855.706 | 5835.306 | 5814.906 |

# References


1. *Dimitri P. Bertsekas* Nonlinear Programming. Athena Scientific, Belmont MA, 2016. — 777 C.
2. *Danskin J.* The theory of Max-Min and its application to weapons allocation problems. // N.Y. Springer. 1967.
3. *Fiacco A.V., McCormick G.P.* Nonlinear Programming: Sequential Unconstrained Minimization Techniques. // N.Y. : John Wiley and Sons, 1968. – 210 c.
4. *Fiacco A.V.* Introduction to Sensitivity and Stability Analysis in Nonlinear Programming. // N.Y. : Academic Press, 1983.
5. *Fudenberg Dr., Tirole J.* Game Theory. The MIT Press. : Cambridge, Massachusetts, 1991. – 604 P.
6. *Rockafellar T.* Convex Analysis. // Prinseton University Press. 1970. v.3.
7. *Umnov A.E., Umnov E.A.* Feedback Functions in Problems of Nonlinear Programming. // SCIREA Journal of Mathematics, 2022, Volume 7, Issue 5, PP 67-82, DOI: 10.54647/mathematics11348
8. *Umnov A.E., Umnov E.A.* Using feedback functions to solve parametric programming problems // Computer Research and Modeling, 2023, vol. 15, no. 5, pp. 1125-1151 DOI: 10.20537/2076-7633-2023-15-5-1125-1151
9. *Ашманов С.А.* Линейное программирование. М. : Наука, 1981. — 340 С.
10. *Гермейер Ю.Б.* Введение в теорию исследования операций. М. : Наука, 1971. – 384 С.
11. *Гермейер Ю.Б.* Приближенное сведение с помощью штрафных функций задачи определения максимина к задаче определения максимума. // ЖВМ-МФ. 1969. Т. 9, № 3. С. 730-731.

12. *Гольштейн Е.Г., Третьяков Н.В.* Модифицированные функции Лагранжа. // М., Наука, 1989. — С. 400.

13. *Демьянов В.Ф., Васильев Л.В.* Недифференцируемая оптимизация. // М. Наука. 1981, 384 С.

14. *Демьянов В.Ф., Малоземов В.Н.* Введение в минимакс. // М., Наука, 1972. — С. 368.

15. *Жадан В.Г.* Методы оптимизации. Ч.2. Численные алгоритмы. §7.2, М.: МФТИ, 2015. – 320 С.

16. *Измаилов А.Ф.* Чувствительность в оптимизации. – М.: Физматлит, 2006. – 245 С.

17. *Кудрявцев Л.Д.* Курс математического анализа (в двух томах). Т. II. М. : Высшая школа, 1981. — 584 С.

18. *Лотов А.В., Поспелова И.И.* Многокритериальные задачи принятия решений. М.: МАКС Пресс, 2008. – 197 С.

19. *Нурминский Е.А.* Численные методы выпуклой оптимизации. М.: Наука, 1991. – 168 С.

20. *Сухарев А.Г., Тимохов А.В., Федоров В.В.* Курс методов оптимизации. [ Электронный ресурс ]. М. : Физматлит, 2011. – 384 С.

21. *Умнов А.Е.* Многошаговая линейная экстраполяция в методе штрафных функций // ЖВМ-МФ. 1974. Т. 14, № 6. – С. 1451–1463.

22. *Умнов Е.А., Умнов А.Е.* Метод параметрической линеаризации, использующий штрафные функции со всюду обратимой производной для решения пар двойственных задач // Труды МФТИ. 2011. Т. 3, № 1(9). С. 146—152.

23. *Умнов Е.А., Умнов А.Е.* Параметрические задачи в математическом программировании. М.: МФТИ, 2018. 297 С. // www.umnov.ru.

2x. *Умнов Е.А.* Параметрические семейства линий второго порядка на плоскости. М.: МФТИ, 2025. 36 С.//www.umnov.ru.

24. *Умнов А.Е., Умнов Е.А.* Использование функций обратных связей в задачах линейного программирования // Ж. вычисл. матем. и матем. физ. 2019. Т. 59, № 10. С. 1681–1694.

25. *Умнов А.Е., Умнов Е.А.* Использование функций обратных связей для решения задач параметрического программирования // Компьютерные исследования и моделирование, 2023, т. 15, № 5, с. 1125-1151 DOI: 10.20537/2076-7633-2023-15-5-1125-1151

26. *Фаддеев Д.К.* Лекции по алгебре. М. : Наука, 1984. 416 С.

27. *Федоров В.В.* Численные методы максимина. // М. Наука. 1979. 279 С.

28. *Федорюк М.В.* Метод перевала. // М. Наука. 1977. 368 С.

29. *Фиакко А., Мак-Кормик Г.* Нелинейное программирование. Методы последовательной безусловной минимизации. М. : Мир, 1972. — 240 С.